Physics of elliptical reflectors at large reflection and divergence angles II: Analysis of optical beam distortions in integrated ultra-large-angle elliptical curved reflectors

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In this work, we study the behavior of elliptical reflectors in nanophotonic integrated circuits in the context of two-dimensional multimode Gaussian beam. We show that the transformation of the beam profile at the input/output waveguide mouths, and the transformation of the beam profile at the reflector can be described in terms of 2D Hermite-Gaussian (HG) beam modes under first order approximation (FOA), referred to as HG-FOA. Due to the wavelength-scale waveguides in nanophotonic integrated circuits, the beams incident on the elliptical reflector have large diffraction angle which will result in asymmetric amplitude distortion upon non-normal reflection even for ideal elliptical reflecting surfaces. The amplitude distortion can be illustrated in the oscillation of the peaks of the reflected beam profiles around the propagation axis of the reflected beam. The amplitude distortion can also be shown by the deterioration in the coupling efficiency from the input waveguide to the output waveguide due to the excitation of higher order HG modes during the reflection. These two observations can both be explained by the HG-FOA method and are verified with Finite-Difference Time-Domain (FDTD) method. Moreover, we show that the coupling loss from the input waveguide to the output waveguide due to the higher order modes excited can be eliminated via insertion of a second reflector. Two specific arrangement of the second reflector are discussed and verified by the FDTD simulation.

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1. Introduction

In nanophotonic integrated circuits (NPIC), the beam transformation cannot be realized easily with integration of traditional refractive lens. Integrated elliptical reflectors provide a convenient way to perform the functions of refractive lenses [1]. In NPIC, light beams usually diffract strongly, which means the beam will have very large diffraction angle of up to over 90° (full diffraction angle). Therefore, to realize off-axis reflection of an incident beam, the use of elliptical-shaped elliptical mirror becomes essential to minimize the phase aberration. Whereas less well known is that when a large angle beam is reflected from the elliptical mirror at an angle (i.e. at off-normal incidence), the beam intensity profile can be distorted [2]. It turns out that the phase front can also be distorted even with use of an elliptical reflector due to Physical Optics Correction [3], but to a much smaller extent. Actually, the intensity profile distortion can be expected even in classical optics when a beam is reflected from an elliptical mirror at a large incident angle. This is because the angular extent of a symmetric beam will become nonsymmetric with respect to the propagation axis (i.e. the angular extent will be different on two sides of the axis) after reflection, which can be easily illustrated in geometric optics. Thus, inevitably, the intensity profile cannot remain as symmetric after reflection from the elliptical reflector. A proper analysis of such intensity profile and phase front distortion is particularly important for implementing large-angle elliptical reflectors. An accurate and commonly used way of study the behavior of the reflector is the Finite-Difference Time-Domain (FDTD) method which is a numerical solution to the Maxwell equations. However it is time-consuming and demands large computational capacity. In comparison, our method based on Hermite-Gaussian (HG) beam mode is analytical and consumes minimal time. We show in this paper that the prediction of our method is in reasonable agreement with the FDTD numerical solution in different applications. Therefore, our method provides a faster yet reliable way for the designing of complex NPIC containing large amount of elliptical reflectors.
A main focus of this paper is to derive a set of equations that will enable one to analyze the beam distortion after reflection from an elliptical mirror at large incident angle analytically and accurately. As it turns out, the distortion can be analyzed in the context of two-dimensional HG modes analysis in that the distortion can be understood as due to the excitation of higher-order HG beam modes. We show some examples where our method, under first order approximation (FOA), referred to as HG-FOA method, matches the results of FDTD method. It implies that, although HG beam modes is an excellent approximation only under paraxial propagation approximation, HG-FOA method is still a reasonable approximation even when the diffraction angle is relatively large at around 90° (full angle) diffraction angle. In these examples, we show both analytically that the peaks of the reflected beam profiles can shift from one side to another with respect to the optical axis of propagation, which is supported by the FDTD simulation. This can be understood using the mode decomposition picture where the differential Guoy phase between the symmetric lowest order mode and anti-symmetric first and third order modes causes phase beating. We also show that due to the presence of higher order modes, the transmission efficiency from the input waveguide to the output waveguide is compromised. However, as we will point out, the transmission loss can be eliminated by compensating the distortion via a second reflector with proper configurations. In addition, it is worth noting that, in this paper the derivation is based on planar waveguide optics as opposed to free space optics used in [2]. This leads to similar but fundamentally different solutions of beam behaviors as will be elaborated later in Section 3.

The paper is organized as follows. Section 2 formulates the two-dimensional free space beam propagation using HG beam analysis. The beam amplitude transformation at the waveguide mouth is studied by Gaussian beam decomposition. We point out a good choice of the mode set is such that the power contained in the lowest order HG mode is maximized. Section 3 studies the beam amplitude transformation by an elliptical reflector based on perturbation analysis. Analytical expressions of reflected beam modes are obtained. The result is in turn applied to the elimination of the transmission loss via insertion of a second reflector which potentially cancels out the distortion induced by the reflector. Section 4 shows calculation examples of applying HG-FOA method to calculate the coupling of beam from waveguide to free space, the reflected beam profile in an elliptical reflector, the total transmission efficiency of an elliptical reflector and the distortion compensation via a second reflector. These are verified by FDTD simulation.

2. Amplitude transformation at waveguide mouth

In this section we study the beam coupling between the input (output) waveguide and the two-dimensional free space region occupied by the reflector. We show that the fundamental waveguide mode can be decomposed in terms of a complete HG mode set. The mode set can be conveniently chosen so that the overlapping integral of its lowest order mode and the fundamental waveguide mode is maximized at the waveguide mouth. We only consider the fundamental waveguide mode because it experiences the least propagation loss in practice.

2.1. Derivation of the incident beam mode

In [1] we have described the incident and reflected beams of an elliptical reflector in terms of simple Gaussian beam. Fig. 1 shows the geometry of such a two-dimensional elliptical reflector configuration. In Fig. 1, we assume the phase front of the incident beam has its center of curvature at $O_1$. The propagation axis of the incident beam is $O_1C$ where $C$ is on the reflector. The phase front of the reflected beam has its center of curvature at $O_2$. The propagating axis of the reflected beam is defined by $O_2C$. The angle between the propagating axes of the incident and reflected beams are defined as turning angle $\theta_1$, $O_1(x, y, z)$ coordinate axes are defined such that the $z$-axis lies along $O_1C$, with $z = 0$ at $C$ and $z = -R_{in}$ at $O_1$ (with $z < 0$), where $R_{in}$ is the radius of curvature of phase front of the incident beam at $C$. $O_2(z', y', z')$ coordinate axes are such that $z'$ axis coincides with the optical axis of the reflected beam, $O_2C$, with $z' = 0$ at $C$ and $z' = R_{out}$ at $O_2$ (with $z' > 0$), where $R_{out}$ is the radius of curvature of phase front of the reflected beam at $C$. Subscript $in$ is for variables associated with the input (incident) side, subscript out is for variables associated with the output (reflected) side. The $(z, z')$ plane lies in the plane of the paper in such a way that the $y'$ and $z$ axes coincide. The beam waist radii of the incident and reflected beam are $w_{in}$ and $w_{out}$. The distance from $B_1$ to $C$ is $L_{in}$ and distance from $B_2$ to $C$ is $L_{out}$.

The input waveguide and output waveguide have widths of $D_{in}$ and $D_{out}$. The equations used are summarized as follows:

\[
F(x, Z) = \sum_n c_n \psi_n(x, Z) = \sum_n c_n A_n(x, w(Z)) e^{i\phi_n(x, Z)}
\]

(1)

\[
A_n(x, w(Z)) = \left( \frac{2}{\pi w^2} \right)^{1/4} \frac{1}{2^n n!} H_n^0 \frac{\sqrt{2}}{w(Z)} \exp \left( - \frac{x^2}{w^2(Z)} \right)
\]

(2)

\[
\phi_n(x, Z) = \exp \left[ -jk \left\{ \frac{Z + x^2}{2R(Z)} \right\} \right] + j(n + 1/2) \Delta \phi_0(Z)
\]

(3)

\[
w(Z) = w_0 \sqrt{1 + \left[ \frac{Z}{Z_c} \right]^2}
\]

(4)

\[
R(Z) = Z \left[ 1 + \left( \frac{Z}{Z_c} \right)^2 \right]
\]

(5)

\[
\Delta \phi_0(Z) = \zeta(Z) = \tan^{-1} \left[ \frac{Z_c}{(\pi w_0^2)} \right]
\]

(6)
where \( F(x, Z) \) represents magnetic field \( H_{y}(x, Z) \) for TE mode and electric field \( E_{x}(x, Z) \) for TM mode with \( Z=z-z_{0} \), \( z_{0} \) is the beam waist position. \( \psi_{n}(x, Z) \) is the normalized complex amplitude of the HG mode. \( c_{n} \) is the complex mode coefficient. \( \Delta \phi_{0}(x, Z) \) and \( \phi_{0}(x, Z) \) are the amplitude and phase terms of \( \phi_{0}(x, Z) \). \( H_{y}(s) \) represents a Hermite polynomial of order \( n \) with argument \( s \). \( w(Z) \) is the beam radius of the mode at \( Z \). \( w_{0} \) is the beam waist radius of the set of HG modes. \( \Delta \phi_{0}(Z) \) is the phase slippage which is also called the Guoy phase term \( \zeta(Z) \). \( R(Z) \) is the radius of curvature of the phase front at \( Z \). Notice that the initial phase of the \( n \)-th order mode is set to be zero since it can be absorbed in the complex mode coefficient.

In general, the choice of the mode set to decompose a given light beam is arbitrary since the HG modes are complete sets. From Eqs. (1) and (2), it is seen that a unique set of HG modes is defined by the beam waist radius \( w_{0} \) and beam waist position \( z_{0} \). For an arbitrary combination of \( w_{0} \) and \( z_{0} \), the mode coefficients \( c_{n} \) of the decomposed HG modes are determined by overlapping the total field of the light beam and the corresponding HG mode profile at the beam waist since all modes in a complete mode set are orthogonal. Suppose the total field profile at the beam waist is \( F(x) \), it can be written in the form of HG decomposition

\[
F(x) = \sum_{n} c_{n}A_{n}(x, w_{0}) \tag{7}
\]

where \( z_{0} \) is set to be 0, \( A_{n} \) is given by (2). Note that for the input waveguide, \( F(x) \) is not equal to the fundamental waveguide mode due to reflection and diffraction at the waveguide mouth but has an exact solution defined by the input waveguide. In order to obtain accurate \( F(x) \) we can either use rigorous analytical calculation or use a numerical mode profile at just outside the waveguide mouth. Given \( F(x) \), the HG mode coefficient is calculated as

\[
c_{n} = \frac{\int F(x)A_{n}^{*}(x, w_{0})dx}{\sqrt{\int F^{2}(x)dx \times \int A_{n}^{2}(x, w_{0})dx}} \tag{8}
\]

Note that we have made the 2D effective refractive index approximation so that the field does not change along the \( y \) axis.

In an elliptical reflector, the propagating beam in the free space starts from the input waveguide mouth. Therefore, we should consider the coupling from the input waveguide to the free space so as to determine the HG mode set in the free space. In this case, it is natural to let the HG beam waist to be at the input waveguide mouth since the phase fronts of the HG modes are flat at their waists, which matches the flat phase front of the waveguide modes. In order to determine the beam waist radius \( w_{0} \), a good choice is to choose a \( w_{0} \) such that the lowest order mode of the HG beam has the greatest weight amongst all modes. This is because not only the lowest order HG mode best resembles the fundamental waveguide mode but also it simplifies the derivation of the reflected beam modes as will be shown later. We define the weight of the lowest order mode by its mode coefficient. So maximizing the weight of the lowest order HG mode is equivalent to maximizing the mode coefficient of the lowest order HG mode \( c_{0} \). It can be demonstrated that by maximizing the mode coefficient of the lowest order mode, the power contained in the lowest order mode is also maximized. This is because the power coupling efficiency is defined as

\[
\eta_{\text{eff}} = \frac{\int |\phi(x)\phi^{*}(x)dx|^{2}}{\int |\phi^{2}(x)dx| \times \int |\phi^{*}(x)dx|} \tag{9}
\]

where \( \phi(x) \) and \( \phi^{*}(x) \) are two coupled beam mode profiles. By comparing (7)–(9), it is found that

\[
\eta_{\text{eff}} = |c_{0}|^{2} \tag{10}
\]

It implies that, by maximizing \( c_{0} \), the power coupled into the lowest HG mode is maximized.

At the output waveguide mouth, the reflected beam from the reflector is coupled into the output waveguide. If the mode composition of the reflected beam can be known and the output waveguide is given, the power coupling efficiency can be readily calculated using (9) by replacing \( \phi(x) \) and \( \phi^{*}(x) \) with the reflected HG beam profile and the fundamental waveguide mode profile of the output waveguide. On the other hand, we can choose the output waveguide width to optimize the coupling efficiency via calculating (9). In the following section, we discuss how the reflected beam modes are determined knowing the incident beam.

2.2. Example: Coupling from waveguide to free space

In this example, we show a numerical example to illustrate how to apply above mentioned analysis based on HG beam to a specific input waveguide. The HG modes expression of the incident beam is then determined.

In order to be consistent in this section and Section 3, we use the same example for discussion. The example is described as follows: A TE polarized beam at wavelength 1.55 \( \mu \)m is fed as input beam. The effective refractive index of the waveguide core is \( n_{\text{core}} = 3.2 \), the effective refractive index of the cladding is \( n_{\text{clad}} = 1 \). The input waveguide width is \( W_{\text{in}} = 1 \mu \)m. The output waveguide width is \( W_{\text{out}} = 4.70 \mu \)m. The waveguide turning angle \( \theta = 90^\circ \).

The parameters of the reflector are listed as follows: the major radius \( a = 30 \mu \)m, the minor radius \( b = 15.81 \mu \)m, the angle between the major axis and the optical axis of the input waveguide is \( \theta = 78.69^\circ \). The geometry of the reflector can be solved by following our previous paper [1]. We list the results as follows: The on-axis distance between the incident beam waist and the center of the reflector \( C \) is \( L_{\text{in}} = 9.93 \mu \)m, the on-axis distance between the reflected beam waist and the center of the reflector \( C \) is \( L_{\text{out}} = 42.36 \mu \)m. The radii of curvature of the incident and reflected beams at the reflector are \( R_{\text{in}} = 10 \mu \)m and \( R_{\text{out}} = 50 \mu \)m.

We have shown that the HG modes composition of the output beam of the input waveguide can be calculated using (8). In order to use (8), we write the analytical solution of the fundamental waveguide mode \( u(x) \)

\[
u(x) = \begin{cases} 
E_{1} = C_{1}\exp[-\tau_{1}(x-d/2)] & x > d/2 \\
E_{2} = C_{2}\cos(k_{d}x) & -d/2 < x < d/2 \\
E_{3} = C_{2}\exp[\tau_{2}(x+d/2)] & x < -d/2 
\end{cases} \tag{11}
\]

where \( k_{d} \) is the transverse component of the wave vector, \( \tau_{1} \) and \( \tau_{2} \) are the extinction coefficients on both sides of the claddings. The parameters of the waveguide mode for the 1 \( \mu \)m input waveguide are given by a normal mode solver as: \( k_{d} = 2.70 \text{\mu m}^{-1} \), \( \tau_{1} = \tau_{2} = 12.02 \text{\mu m}^{-1} \). By substituting \( F(x) \) in (8) with the obtained \( u(x) \) and plug in \( A_{0}(x, w_{0}) \) given by (2), and implementing an iteration algorithm in matlab, we can obtain the \( w_{0} \) that maximizes \( c_{0} \). It turns out that \( w_{0} = 0.36 \mu \)m and \( c_{0} = 0.9965 \). Using (10) the power contained in the lowest-order mode is then calculated to be \( \eta_{\text{eff,analytic}} = |c_{0}|^{2} = 99.30\% \). Thus the lowest order mode contains almost all the power. It is convenient to assume that the incident beam to the reflector is only the lowest order HG mode.

However, \( F(x) \) is not exactly equal to \( u(x) \). To be accurate, we can use numerical field profile \( F(x) \) right outside the waveguide mouth. The numerical field profile is obtained by the FDTD method. The waveguide structure used in the FDTD method is shown in Fig. 2a and b is the snapshot of the generated \( H_{y} \) field. The filed profile \( F(x) \) is acquired by taking the field profile just one pixel outside (right side of waveguide mouth in Fig. 2b), which is plotted in Fig. 6a in circles. By substituting \( F(x) \) with the
obtained $F_{\text{sim}}(x)$ in (8), we can find that $w_0 = 0.353 \mu m$ and $c_0 = 0.9957$. The power contained in the lowest order mode is then calculated by (10) to be $\eta_{\text{eff, numerical}} = |k_0|^2 = 99.15\%$. The corresponding lowest order HG mode is shown in dashed line in Fig. 6a. Comparing the analytical result to numerical result, the difference in $w_0$ is less than 2%, the difference in power ratio is less than 0.2%, suggesting the analytical method is a good enough estimation. Therefore, in this paper, we use the analytical method for simplicity.

3. Amplitude transformation at the reflector

The aim of this section is to study the behavior of the reflector using the Hermite-Gaussian (HG) beam mode analysis. We show that the transformation of the incident beam by a reflector can be represented by a scattering matrix. The reflected beam turns out to be consisted of the lowest, first and third order HG modes. We also show that one application based on this finding is using a second reflector to compensate the distortion due to higher order modes and achieve nearly perfect transmission from the input waveguide to the output waveguide fundamental mode.

3.1. Ray picture of the distortion

In [1] we pointed out that the phase distortion in an elliptical reflector is negligible. However the amplitude distortion is inevitable in the reflected beam. This is illustrated in ray picture in Fig. 3. An imaginary input beam has a full diffraction angle of 64̊ defined by the angle sustained by the propagation locus of the 1/e point of the field amplitude. Suppose the input rays originate from one of the focus of the reflector: $O_1$. Following the ray picture, we draw the reflected rays by connecting incident points of input rays on the reflector with the other focus $O_2$. It is seen that the diffraction angle of the reflected rays: 89̊, illustrated by the caustic, is larger than that of the input beam. More importantly, the reflected rays are asymmetric with respect to the propagation axis $OC_2$. Specifically, the angles between the caustic and propagation axis are 21̊ and 68̊. This indicates the uneven distribution of power with respect to the propagation axis, which simply shows that the reflected beam is distorted and shall not be represented by a symmetric lowest order Gaussian mode.

3.2. Derivation of the reflected beam mode

To visualize the reflection in a more intuitive way, we treat the reflected beam as if it is transmitted through the reflector. This is done by reflecting the actual reflected beam and the surface of the reflector in the plane tangent to the surface of the reflector at $C$. Fig. 4 illustrates the caustic of the incident and reflected beams. Plane $M$-$M'$ is the tangent plane, which has an angle of $\pi/2 - \theta$ to the z-axis. The incident angle at $C$ is $\theta_1 = \theta/2$. The actual reflected beam and the reflector are reflected with respect to plane $M$-$M'$ to form the imaginary scenario of “transmitted” beam. An arbitrary point $q$ on the actual surface of the reflector has an image at $q'$ on the imaginary reflector. Line $qq'$ has an intersection with $M$-$M'$ plane at $p$. It is easily seen that the beam size of the input beam and reflected beam at the same point on the reflector, such as $q$ and $q'$, are different. This suggests that there is distortion in the reflected beam. In the following context, we express the amplitude distortion as a first-order perturbation expansion of the lowest order HG mode of the reflected beam. The treatment here follows the treatment of [2]. Note that we have treated the planar waveguide as 2D waveguide, which is fairly reasonable because, for planar waveguide, in the vertical direction light is confined in the core layer, while in the horizontal direction light is free to diffract. This is distinct from free space optics in [2] where the spatial mode in the x and y directions are associated. Consequently this leads to different mathematical forms of the reflected beam solutions as will see below. In general, using [1], the incident beam and reflected beam can be described in terms of HG modal expansion. For convenience, we assume the incident beam solutions as...
Therefore, we write

\[ A_m(x_r, w_r) = A_m(x_r + \Delta x, w_r + \Delta w) = A_m(x_r, w_r) + \left( \frac{\partial A_m}{\partial x} \right) (x_r, w_r) \Delta x + \left( \frac{\partial A_m}{\partial w} \right) (x_r, w_r) \Delta w \]  

(18)

where \( \Delta x \) and \( \Delta w \) are given by (15) and (17). By plugging the expressions of \( A_m(x, w) \), \( \Delta x \) and \( \Delta w \), in the second and third terms on the right side of (18) and compare (18) to (13), we can write the reflected beam in the form as scattering matrix (see Appendix)

\[ A(x_r, w_r) = \sum_k S_{m,k} A_k(x_r, w_r) \]  

(19)

where \( S_{m,k} = \delta_{m,k} + \alpha_{m,k} \), \( S_{m,1} = 1 \) when \( m \neq k \). For \( m \neq k \), the scattering matrix is given by

\[ S_{m,k} = \left( \frac{w_m \tan \theta_i}{8f} \right) \times \begin{cases} \sqrt{(m+1)(m+2)(m+3)} & k = m + 3 \\ \sqrt{(m+1)^2} & k = m + 1 \\ -\sqrt{m^3} & k = m - 1 \\ -\sqrt{(m-2)(m-1)m} & k = m - 3 \end{cases} \]  

(20)

which only relates to the incident beam mode orders \( m \), the reflected beam mode order \( k \), focal length \( f \), turning angle \( \theta \), and beam radius at the reflector \( w_m \). (20) means that after reflection the power in the reflected beam is scattered to neighboring modes of the incident beam, specifically, \( m - 3 \), \( m - 1 \), \( m + 1 \), \( m + 3 \) modes. We have shown in Section 2.2 that, in our case, the lowest order HG mode contains almost all the power in the incident beam. Thus we can conveniently assume the incident beam is the lowest order HG mode only. This means \( m = 0 \). Therefore, in the reflected beam the power is scattered into \( k = 1 \) and \( k = 3 \) modes.

\[ A(x_r, w_r) = A_0(x_r, w_r) + S_1 A_1(x_r, w_r) + S_3 A_3(x_r, w_r) \]  

(21)

where, by using (20) we have

\[ S_1 = \frac{w_m \tan \theta_i}{8f} \]  

(22)

\[ S_3 = \sqrt{6w_m \tan \theta_i}/(8f) \]  

(23)

where we have omitted the first part in the subscripts of \( S_{0,1} \) and \( S_{0,3} \) for simplicity. It is worth pointing out that this solution is different from the three-dimensional solution. In the three-dimensional solution given in [2], for a fundamental mode as input, in the reflected beam, the power is scattered to (30) and (12) modes with scattering matrix coefficient \( S_{0,30} = \sqrt{6w_m \tan \theta_i}/(8f) \) and \( S_{0,12} = \sqrt{2w_m \tan \theta_i}/(8f) \), which indicating different power ratio distribution in two higher modes. With (21), we conclude that if we use an output waveguide to couple the reflected beam out, only the power contained in the lowest order HG mode can be coupled into the output waveguide. This is because the first and third order HG modes are asymmetric, which cannot couple into a symmetric fundamental waveguide mode.

### 3.3. Distortion compensation via a second reflector

The existence of the asymmetric first and third order modes in the reflected beam will results in direct coupling of reflected beam into a fundamental waveguide mode is inherently lossy. However, in this section we show that by implementing a second reflector we can eliminate the transmission loss in the output waveguide. This can be shown by studying the reflected beam propagation equation. Since the reflected beam only contains the lowest, first and third order mode, as shown in Appendix the
reflected beam can then be written as

\[ F_i(x, z) = A_0(x, w_i(z)) e^{i\phi(x, z)} + g_1(x_0, w_i(z)) S_1 A_1(x, w_i(z)) e^{i\phi(x, z)} + g_2(x_0, w_i(z)) S_2 A_2(x, w_i(z)) e^{i\phi(x, z)} \]

For configurations where the beam propagation is reversible, if we reflect the beam and re-reflect it to an output waveguide, we should have two special cases that are worth noting. The first case is when

\[ g_1(x_0, w_i(z)) = g_2(x_0, w_i(z)) = 0 \]

Eq. (24) can then be written as

\[ F_i(x, z) = A_0(x, w_i(z)) + S_1 A_1(x, w_i(z)) + S_2 A_2(x, w_i(z)) \]

where we have omitted the shared phase term \( g_0(x, z) \) since it does not change the amplitude profile. In this case, the amplitude profile denoted by (26) is the same as amplitude profile at the reflector denoted by (21) except the beam radius \( w_i \) is a variable. Whereas the second case is when

\[ g_1(x_0, w_i(z)) = g_2(x_0, w_i(z)) = 1 \]

Eq. (24) is then written as

\[ F_i(x, z) = A_0(x, w_i(z)) - S_1 A_1(x, w_i(z)) - S_2 A_2(x, w_i(z)) \]

where we have omitted the shared phase term \( g_0(x, z) \) as well. In (28), the higher order components have opposite signs as compared to the amplitude profile at the reflector denoted by (21).

Now consider a second reflector is inserted to collect the reflected beam and re-reflect this beam to an output waveguide that is the same as the input waveguide. For a single elliptical reflector, because the beam propagation is reversible, if we reverse the beam propagation and have the previous reflected beam as input beam, we shall have a reflected beam that is the same as the previous input beam. This requires the beam radius at the input waveguide to be infinitely small. This is to say that, the reflected beam of the first reflector should be focused to a point source, which essentially becomes the geometrical coordinates if (25) is satisfied. This requires

\[ \Delta \phi_0(Z) + \Delta \phi_0(L_{out}) = \zeta(Z) + \zeta(L_{out}) = 2n\pi \]

Since we have \(-\pi/2 \leq \zeta(x) \leq \pi/2\), and \(Z_r \geq L_{out}\), this implies \(0 \leq \zeta(Z_r) + \zeta(L_{out}) \leq \pi\). So (29) becomes

\[ \Delta \phi_3(Z_r) + \Delta \phi_0(L_{out}) = \zeta(Z_r) + \zeta(L_{out}) = 0 \]

which requires both \(\zeta(L_{out})\) and \(\zeta(Z_r)\) are zero since both \(L_{out}\) and \(Z_r\) are positive. This implies that \(\zeta(L_{out}) = L_{out}/(\pi w_{out}^2 / \lambda) = 0\), and \(\zeta(Z_r) = Z_r/(\pi w_{out}^2 / \lambda) = 0\). Actually, the reflected beam in this case is collimated and shall have uniform beam profile along its propagation path. Since the \(z\) axis points to the same direction as the \(x\) axis, the incident beam on the second reflector is in turn the same as the reflected beam on the first reflector. Therefore, if the output waveguide is the same as the input waveguide, the coupling of reflected beam from the second reflector to the output waveguide should be perfect. It is worth noting that, design-wise, we should make the \(R_{out}\) infinitely large compared to \(L_{out}\) so as to implement configuration in Fig. 5a. \(R_{out}\)

In the configuration shown Fig. 5b, a second reflector is of the same shape as the first reflector and is point symmetric to the first reflector with respect to the center of reflected beam waist. Similar to the previous configuration, we defined the \(x' - z'\) axes affixed to the second reflector as shown. It can be seen that, if the incident beam on the second reflector can be expressed as (28) in the coordinates affixed to the first reflector (\(x - z\) axes), this incident beam can be expressed as (21) in \(x - z\) axes. Because the two reflectors are the same, the coupling of reflected beam from the second reflector to the output waveguide should be perfect. In order that the incident beam on the second reflector can be expressed as (28), it requires (27) to be satisfied, which in turns requires the phase terms

\[ \Delta \phi_0(Z_r) + \Delta \phi_0(L_{out}) = \zeta(Z_r) + \zeta(L_{out}) = 2(n + 1)\pi \]

Similar to the previous case, we have \(0 \leq \zeta(Z_r) + \zeta(L_{out}) \leq \pi\). So (31) becomes

\[ \Delta \phi_0(Z_r) + \Delta \phi_0(L_{out}) = \zeta(Z_r) + \zeta(L_{out}) = \pi \]

which requires \(\zeta(Z_r) = -\zeta(L_{out}) = \pi/2\), meaning \(Z_r = L_{out}\) and \(L_{out}/(\pi w_{out}^2 / \lambda) \rightarrow \infty\). Because \(L_{out}/(\pi w_{out}^2 / \lambda)\) is infinitely large and \(L_{out}\) should be finite, it requires \(w_{out}\) to be infinitely small. This is to say that, the reflected beam of the first reflector should be focused to a point source, which essentially becomes the geometrical

![Fig. 5](image-url) Illustration of two scenarios of distortion compensation. (a) 360° turn scenario. (b) 180° turn scenario.
optics scenario. Again, in terms of design, to realize such scenario, it requires $L_{\text{out}} = R_{\text{out}}$.

3.4. Examples

In this section, we show some numerical examples to illustrate how to apply above mentioned analysis to specific reflector configurations. This includes the determination of HG modes expression of the reflected beam, how to use the obtained reflected beam expression to predict the evolution of the profile peaks, how to calculate the total transmission efficiency from the input waveguide to the output waveguide and how to use a second reflector to realize optimal total transmission efficiency. The same waveguide and reflector structure as in Section 2.2 is used for consistency.

3.4.1. Example 1: Calculation of the reflected beam profile

Because the incident beam is the lowest order HG mode, the expression of the reflected beam is given by (24). Note that the focal length of the reflector is $f=1/(1/R_{\text{in}}+R_{\text{out}}) = 0.833 \mu m$. $w_{\text{in}}$ can be calculated by plugging $w_{\text{in}}$ and $L_{\text{in}}$ in (4), which gives $w_{\text{in}} = 4.24 \mu m$. By plugging $f$ and $w_{\text{in}}$ in (22) and (23), we obtain $S_1 = 0.0452, S_3 = 0.1106$. Since $L_{\text{out}}$ is given, we can calculate the amplitude profile at any position $Z$. Specifically, let us look at the reflected beam waist where $Z_r = 0$. Using (24)

$$F_r(x_r,0) = \psi_0(x_r,0) + 0.0541x_r\psi_0(x_r,0)e^{i\Delta \psi_0/2} + (0.0774x_r^3 - 0.1618x_r)$$

$$+ [(2/1.67\pi)^{1/4}e^{-x_r^2/1.67^2}e^{j\Psi_0}](42.36/17.99)$$

$$+ [(2/1.67\pi)^{1/4}e^{-x_r^2/1.67^2}e^{j\Psi_0}](42.36/17.99),$$

where we have calculated the Hermite polynomials $H_1$ and $H_2$ as $H_1(\sqrt{2}x_r/w_{\text{out}}) = 1.694x_r$ and $H_2(\sqrt{2}x_r/w_{\text{out}}) = 4.858x_r^3 - 10.162x_r$. And the lowest order mode $\psi_0(x_r, Z_r)$ is given by (1) and can be written as

$$\psi_0(x_r, Z_r) = [2/\pi w^2_r(Z_r)]^{1/4}\exp[-x_r^2/w^2_r(Z_r)]\exp[-jk(Z_r - jkZ_r^2)/(2RZ_r)]$$

$$+ [j(1/2)c_3(Z_r)].$$

The reflected beam waist profile given by (33) is plotted in Fig. 6d. Together, the lowest order, first order and third order modes are plotted separately. To compare, we include the numerical beam waist profile by FDTD. A good agreement between the analytical and numerical results can be observed. Similarly, we plot the reflector beam profile at the reflector where $Z_r = -L_{\text{out}}$, together with the three orders of components. This is shown in Fig. 6c. The numerical result is not plotted because of interference between the incident and reflected beam in the region.
occupied by the reflector. To complete the beam propagation picture, we also plot the analytical and numerical beam profile of the incident beam at the beam waist in Fig. 6a, as well as the analytical beam profile of the incident beam at the reflector in Fig. 6b. This is done by letting the incident beam profile \( F(x_i, Z_i) = \psi_i(x_i, Z_i) \): At the incident beam waist \( Z_i = -L_{in} = 9.93 \, \mu m \), at the reflector \( Z_r = 0 \).

The reflected beam profile at any arbitrary position \( Z_r \) can be calculated similar to the procedures above. It turns out that the peaks of the reflected beam profiles oscillate around the propagation axis. This is the result of the beating between the Guoy phase-terms of the different order Hermite-Gaussian modes. Fig. 7a shows how the peaks indicated by (24) evolve along the propagation axis starting from the reflector. It turns out that when \( Z_r \) is large, the peak is shifting linearly in the transverse direction. This is because for large \( Z_r \), the Guoy phase term \( \zeta(Z_r) \) becomes a constant: \( \pi/2 \) for \( Z_r > 0 \), \( -\pi/2 \) for \( Z_r < 0 \), hence no beating happens. Fig. 7b shows how the analytical results compared with the FDTD simulation results in terms of the beam profile peaks. Two results are found to agree reasonably well. Four exemplary field profiles at different positions along the propagation axis of the reflected beam are illustrated in Fig. 8. The four positions are at \( Z_r = -19.935 \, \mu m, -10.035 \, \mu m, 19.915 \, \mu m, \) and \( 39.94 \, \mu m \), which are marked in the insets of \( H_0 \) field snapshot. Discernible shifting in the profile peak can be observed in these figures and the shifting follows the prediction in Fig. 7b.

### 3.4.2. Example 2: Calculation of the total coupling efficiency

The total coupling efficiency of an elliptical reflector from the input waveguide to the output waveguide can be calculated by knowing the coupling efficiencies at the input and output waveguide mouth as well as the transmission efficiency at the reflector. We have calculated the coupling efficiencies between the lowest order HG mode and the corresponding fundamental waveguide mode for the incident beam, which is \( n_{eff,inc} = 99.3% \). Suppose the transmission efficiency at the reflector is unit, we only need to calculate the coupling efficiency at the output waveguide. Since the reflected beam contains the lowest, first and third order HG modes, only the lowest order HG mode can be coupled to the fundamental waveguide mode due to symmetry, the coupling efficiency at the output waveguide can be obtained by knowing the coupling efficiency of the lowest order HG mode and the fundamental waveguide mode, as well as the ratio of power contained in the lowest order HG mode in the reflected beam. The power contained in the lowest order mode of the reflected beam is given by

\[
\eta_0 = \frac{1}{1 + S_1^2 + S_2^2} \approx 1 - S_1^2 - S_2^2
\]

where we have assumed that scattering matrix terms \( S_1 \) and \( S_2 \) are small enough. By plugging \( S_1 = 0.04515 \) and \( S_2 = 0.1106 \), it is found that \( \eta_0 = 98.8% \). Then, by using a mode solver, the fundamental mode parameters of the output waveguide are given as: \( k_d = 0.666 \, \mu m^{-1}, \tau_1 = \tau_2 = 12.304 \, \mu m^{-1} \), where \( k_d \) and \( \tau_1, \tau_2 \) are defined in section 2.2. The coupling efficiency is then calculated by substituting \( \phi(x) \) and \( \phi(x) \) in (9) with the fundamental waveguide mode \( u(x) \) of the output waveguide and the lowest order HG mode \( A_0(x, w_0) \), which turns out to be \( \eta_{eff} = 99.4% \). The total coupling efficiency is then calculated by multiplying \( \eta_{b}, \eta_{eff,inc} \) and \( \eta_{eff,ref} \). This gives the analytical total transmission efficiency \( \eta_{total,analytical} = 99.3\% \times 98.6\% \times 99.4\% = 97.3\% \). In comparison, we found the transmission efficiency from FDTD simulation is \( \eta_{total,simulated} = 97.2\% \). The agreement of our HG-FOA method and the numerical result suggests our method is a reasonable prediction of the reflected beam.

#### 3.4.3. Example 3: Distortion compensation via a second reflector

In section 3.3, we pointed out that configurations utilizing a second reflector to compensate the distortion induced by the first reflector can potentially provide unit total transmission efficiency. The discussion in section 3.3 implies that if the reflected beam of the first reflector is a collimated beam, the 360° turn configuration provides higher total transmission efficiency against the 180° turn configuration. The difference in the total transmission efficiency can be considered to be contributed by higher order HG modes in the reflected beam of the first reflector and the coupling loss of coupling a lowest order HG mode to the output waveguide by the second reflector. It is also implied that if the reflected beam of the first reflector is a focused beam, the 360° turn configuration should provide higher total transmission efficiency for the same reason. We verified these two scenarios by using FDTD method to simulate two structures described as follows:

1. The input and output waveguide widths are \( W_{inc} = 1 \, \mu m \) (corresponding to Gaussian beam waist radii \( w = 0.353 \, \mu m \)). The effective refractive indices of waveguide core and cladding are \( n_{core} = 3.2, n_{clad} = 1 \). The waveguide turning angle is \( \theta = 90° \). \( R_{in} = 10 \, \mu m \) and \( R_{out} = 500 \, \mu m \), \( L_{in} = 9.93 \, \mu m \) and \( L_{out} = 25.90 \, \mu m \). The second reflector is placed such that it intersects the propagation axis of the reflected beam of the first reflector at \( Z_r = L_{out} \).

2. The input and output waveguide widths are \( W_{inc} = 1 \, \mu m \) (corresponding to Gaussian beam waist radii \( w = 0.353 \, \mu m \)). The effective refractive indices of waveguide core and cladding are \( n_{core} = 3.2, n_{clad} = 1 \). The waveguide turning angle is \( \theta = 90° \). \( R_{in} = 10 \, \mu m \) and \( R_{out} = 10 \, \mu m \), \( L_{in} = 9.93 \, \mu m \) and \( L_{out} = 9.93 \, \mu m \). The second reflector is placed such that it intersects the propagation axis of the reflected beam of the first reflector at \( Z_r = L_{out} \).

It is found that in the first structure, the total transmission efficiency is 95.0% for the 360° turn configuration and 92.3% for
the 180° turn configuration; whereas in the second structure the total transmission efficiency is 81.2% for the 180° turn configuration and 92.3% for the 360° turn configuration. The difference in transmission efficiency is in accordance with our prediction. The $H_y$ field snapshots of FDTD simulations for the four structures mentioned above are shown in Fig. 9. In sum, we show that the distortion compensation scheme works as expected.

4. Conclusion

We have shown that in two-dimensional nanophotonic integrated circuits (NPICs) the amplitude transformation of a propagating beam in an elliptical reflector structure, both at the waveguide mouths and on the reflector, can be represented and analyzed via two-dimensional Hermite-Gaussian beam modes analysis under...
first order approximation (HG-FOA). The reflected beam profiles
and profile peak positions predicted by this analysis are in
agreement with FDTD simulation results. The Application of such
analysis in elimination of transmission loss using a double-
reflector scheme is also studied and verified by FDTD simulations.
The proposed HG-FOA analysis provides a simple and accurate
prediction of on-chip beam propagation therefore enables us to
design complex Nano-Photonic Integrated Circuits where large
amount of elliptical reflector could be implemented.

Appendix

Derivation of an expression for $\Delta x$

Now we derive the expression for $\Delta w$ and $\Delta x$. It turns out it is
more convenient to write (4) as

$$w^2(x,z) = w_m^2 \left( 1 + \left[ (L_m + z)/(\pi w_m^2/\lambda) \right]^2 \right) = w_m^2 + 2 L_m z/(\pi w_m^2/\lambda)^2 + \left( 2\lambda/(\pi w_m) \right)^2$$

for incident beam, and

$$w^2(x,z) = w_m^2 \left( 1 + \left[ (L_m - Z_{out})/(\pi w_m^2/\lambda) \right]^2 \right)$$

$$= w_m^2 - 2 L_m Z_{out}/(\pi w_m^2/\lambda)^2 + \left( 2\lambda/(\pi w_m) \right)^2$$

for reflected beam, where we have chosen the incident and
reflected beam radii at the reflector to be equal. By noting that
$\pi w_m^2/(\pi w_m^2) = \lambda L_m/(\pi w_m^2)$, and $\pi w_m^2/(\pi w_m^2) = \lambda L_m/(\pi w_m^2)$, we have

$$w^2(x,z) = w_m^2 \left( 1 + 2 \pi / R_m + z^2/(L_m R_m) \right)$$

(38)

$$w^2(x,z) = w_m^2 \left( 1 + 2 \pi / R_{out} + z^2/(L_m R_{out}) \right)$$

(39)

Consider the incident and reflected beam radii at p, we can write

$$w^2 - w^2 = 2 \left[ \frac{Z}{Z_{out}} \right] + \frac{Z_{out}^2}{Z_{out}^2} \left[ \frac{Z}{Z_{out}} \right]^2 \left( \frac{Z}{Z_{out}} \right)^2$$

(40)

Assuming that $\Delta w \ll w_m \approx w_i \approx w_r$, $\Delta w$ is approximated as

$$\Delta w \approx Z \left( \frac{1}{Z_{out}} \right) w_m + \left( \frac{Z}{Z_{out}} \right)^2 \left( \frac{Z}{Z_{out}} \right)^2$$

(41)

Since on the tangent plane, $z = x \tan \theta$, we can express $\Delta w$ as

$$\Delta w = \Delta w \approx x (w_m \tan \theta \pi)$$

(42)

whereas, consider point q and p for the incident beam, we can write

$$w^2(x_q, Z_q) - w^2(x_p, Z_p) = w_m^2 \left[ 2 \Delta z / (R_m + 2 \zeta / (L_m R_m)) \right]$$

(43)

Assuming that $\Delta w \approx \Delta w \approx w_m \approx w_r$, $\Delta w$ is approximated as

$$\Delta w \approx \Delta w \approx \Delta w \approx \Delta w \approx \Delta w \approx \Delta w \approx \Delta w \approx \Delta w \approx \Delta w \approx \Delta w$$

(44)

because $\Delta w \approx \Delta w \approx \Delta w \approx \Delta w$. Similarly, for the reflected
beam

$$\Delta w \approx \Delta w \approx \Delta w \approx \Delta w \approx \Delta w$$

(45)

Then, assuming $\Delta z - \Delta z = \Delta z/2$, $\Delta w - \Delta w$, can be written as

$$\Delta w = \Delta w = \Delta w = \Delta w = \Delta w = \Delta w = \Delta w = \Delta w = \Delta w$$

(46)

Since on the tangent plane, $z = x \tan \theta$, we can express $\Delta w$ as

$$\Delta w = \Delta x (w_m \tan \theta \pi)$$

(47)

Similarly,

$$\Delta w = \Delta x (w_m \tan \theta \pi)$$

(48)

Because $\Delta x \ll x$, $\Delta w_{qp} = \Delta w_{qp} \ll \Delta w_p$, we conclude that

$$\Delta w = \Delta w_{qp} + \Delta w_{qp} + \Delta w_{qp} = x (w_m \tan \theta \pi)$$

(49)

For $\Delta x$, as the surface of the reflector is elliptical, for a point on
the surface via simple geometry, we can write

$$2a = R_m + R_{out} = \sqrt{x^2 + (R_m + z)^2} + \sqrt{x^2 + (R_{out} + z)^2}$$

(50)

Therefore, by assuming $R_m$ and $R_{out}$ are much greater than the
dimension of the reflector, we can write

$$z_{r} = z_{r} \approx \sqrt{x^2 + (2R_m + z)^2} + \sqrt{x^2 + R_{out} + z)^2}$$

(51)

It can be found from Fig. 4 that $(x_i - x) = (z_{r} - z_{r}) \tan \theta$. Therefore

$$\Delta x = x^2 \tan \theta / 2f$$

(52)

Derivation of an expression for the perturbation terms of the reflected beam and the scattering matrix

By plugging expressions of $\Delta w(x, w_i)$, $\Delta x$ and $\Delta w_{p}$ in the second
and third terms on the right side of (18), the two terms can be written as

$$\frac{\partial A_m}{\partial w} \left( x_i, w_i \right) \Delta w = x \left( w_m \tan \theta \pi / w_p \right) \frac{\partial}{\partial w} \left( \frac{2}{\pi} \right), \left( \frac{1}{w_i} \right)^{1/4} 1 \left( \frac{1}{w_i} \right)^{1/4} \sqrt{2} \left( \frac{1}{w_i} \right)^{1/4}$$

(53)

$$\frac{\partial A_m}{\partial x} \left( x_i, w_i \right) \Delta x = \left( \frac{x \tan \theta \pi}{w_p} \right) \frac{\partial}{\partial x} \left( \frac{2}{\pi} \right), \left( \frac{1}{w_i} \right)^{1/4} 1 \left( \frac{1}{w_i} \right)^{1/4} \sqrt{2} \left( \frac{1}{w_i} \right)^{1/4}$$

(54)

Comparing (18) to (13) we know that the sum of (53) and (54)
accounts for the perturbative terms in (13), which can be expanded as

$$\frac{\partial A_m}{\partial w} \left( x_i, w_i \right) \Delta w + \frac{\partial A_m}{\partial x} \left( x_i, w_i \right) \Delta x$$

$$= \left( w_m \tan \theta \pi / w_p \right) \frac{2}{\pi}^{1/4} \left( \frac{1}{w_i} \right)^{1/4} \sqrt{2} \left( \frac{1}{w_i} \right)^{1/4} \exp \left( \frac{x^2}{w_i^2} \right) \times \left( \frac{1}{w_i} \right)^{1/4} \sqrt{2} \left( \frac{1}{w_i} \right)^{1/4}$$

(55)

where we have used the usual recursion relationship for $H_n$

$$2sH_n(s) = H_{n+1}(s) + 2nH_{n-1}(s) + 4s^2H_n(s) = H_{n+1}(s) + 2(2n+1)H_n(s) + 4n(n-1)H_{n-2}(s) + 4n(n+1)H_{n-1}(s) + 8n(n+1)H_n(s)$$

(56)

From (55) we can find the scattering matrix $S_{m,k}$. It is obvious
that $S_{m,k} = 1$ when $m = k$. When $m \neq k$, the scattering matrix is
given by

$$S_{m,k} = S_{m,k} = \left( \frac{w_m \tan \theta \pi}{w_p} \right)$$


\begin{equation}
\left\{ \begin{array}{ll}
v(m+1)(m+2)(m+3) & k = m+3 \\
v(m+1) & k = m+1 \\
-m^2 & k = m-1 \\
-(m-2)(m-1) & k = m-3 \\
\end{array} \right. \tag{57}
\end{equation}

Derivation of an expression for the reflected beam profile at arbitrary positions

Since the reflected beam only contains the lowest, first and third order mode, by using (1), the reflected beam can then be written as

\[ F_r(x_r, Z_r) = c_0 \psi_0(x_r, Z_r) + c_1 \psi_1(x_r, Z_r) + c_3 \psi_3(x_r, Z_r) \tag{58} \]

By plugging (2) in (58), with some manipulation, we can write

\[ F_r(x_r, Z_r) = c_0 A_0(x_r, w_r(Z_r)) e^{i \phi_0(x_r, Z_r)} + c_1 A_1(x_r, w_r(Z_r)) e^{i \phi_1(x_r, Z_r)} + c_3 A_3(x_r, w_r(Z_r)) e^{i \phi_3(x_r, Z_r)} \]

At the reflector, where \( Z_r = -L_{out} \), \( F_r(x_r, Z_r) \) can be written as

\[ F_r(x_r, -L_{out}) = c_0 A_0(x_r, w_r(-L_{out})) e^{i \phi_0(x_r, -L_{out})} + c_1 A_1(x_r, w_r(-L_{out})) e^{i \phi_1(x_r, -L_{out})} + c_3 A_3(x_r, w_r(-L_{out})) e^{i \phi_3(x_r, -L_{out})} \tag{60} \]

On the other hand, at the reflector, the beam profile given by (60) should be equal to the beam profile given by (21). This implies

\[ c_0 = e^{-i \phi_0(x_r, -L_{out})} \tag{61} \]

\[ c_1 = S_1 e^{-i \phi_0(x_r, -L_{out}) + \Delta \phi_0(-L_{out})} = c_0 S_1 e^{-i \phi_0(x_r, -L_{out})} \tag{62} \]

\[ c_3 = S_3 e^{-i \phi_0(x_r, -L_{out}) + 3 \Delta \phi_0(-L_{out})} = c_0 S_3 e^{-i \phi_0(x_r, -L_{out})} \tag{63} \]

For the propagation of the reflected beam, only the relative phase between modes matters. Therefore we can assume \( c_0 = 1 \) and write the mode coefficients of the first and third order modes

\[ c_1 = S_1 e^{-i \phi_0(-L_{out})} = S_1 e^{i \phi_0(L_{out})} \tag{64} \]

\[ c_3 = S_3 e^{-i \phi_0(-L_{out})} = S_3 e^{i \phi_0(L_{out})} \tag{65} \]

If we plug (64) and (65) in (59), we can write the reflected beam as

\[ F_r(x_r, Z_r) = A_0(x_r, w_r(Z_r)) e^{i \phi_0(x_r, Z_r)} + S_1 e^{i \phi_0(L_{out})} A_1(x_r, w_r(Z_r)) e^{i \phi_1(x_r, Z_r)} + S_3 e^{i \phi_0(L_{out})} A_3(x_r, w_r(Z_r)) e^{i \phi_3(x_r, Z_r)} \]

\[ = A_0(x_r, w_r(Z_r)) e^{i \phi_0(x_r, Z_r)} + \sqrt{2} H_1 \frac{\Delta \phi_0(L_{out})}{w_r(Z_r)} + \Delta \phi_0(Z_r) S_1 A_1(x_r, w_r(Z_r)) e^{i \phi_1(x_r, Z_r)} + \sqrt{2} H_3 \frac{\Delta \phi_0(L_{out})}{w_r(Z_r)} S_3 A_3(x_r, w_r(Z_r)) e^{i \phi_3(x_r, Z_r)} \]

\[ = \psi_0(x_r, Z_r) + \sqrt{2} H_1 \frac{\Delta \phi_0(L_{out})}{w_r(Z_r)} \psi_1(x_r, Z_r) + \Delta \phi_0(Z_r) \psi_1(x_r, Z_r) + \sqrt{2} H_3 \frac{\Delta \phi_0(L_{out})}{w_r(Z_r)} \psi_3(x_r, Z_r) \tag{66} \]

where we have let \( c_0 = 1 \).

References