AN ACTIVE-SET METHOD FOR QUADRATIC PROGRAMMING
BASED ON SEQUENTIAL HOT-STARTS

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Abstract. A new method for solving sequences of quadratic programs (QPs) is presented. For each new QP in the sequence, the method utilizes hot-starts that employ information computed by an active-set QP solver during the solution of the first QP. This avoids the computation and factorization of the full constraint and Hessian matrices for all but the first problem in the sequence. The proposed algorithm can be seen as an extension of the iterative refinement procedure for linear systems to QP problems, coupled with the application of an accelerated linear solver method that employs hot-started QP solves as a preconditioner. Local convergence results are presented. The practical performance of the proposed method is demonstrated on a sequence of QPs arising in nonlinear model predictive control and during the solution of a set of randomly generated nonlinear optimization problems using sequential quadratic programming. In these experiments, the method proves to be fairly reliable, despite the lack of global convergence guarantees. The results also show a significant reduction in the computation time for large problems with dense constraint matrices, as well as in the number of matrix-vector products.

Key words. nonlinear programming, quadratic programming, active set, hot-starts, iterative linear solver, preconditioner, sequential quadratic programming, nonlinear model predictive control

AMS subject classifications. 90-08, 90C20, 90C26, 90C30

DOI. 10.1137/130940384

1. Introduction. We are concerned with the solution of quadratic programs (QPs) of the form

\begin{align}
\min_{d \in \mathbb{R}^n} & \quad \frac{1}{2}d^T W d + g^T d \\
\text{s.t.} & \quad Ad + c = 0, \\
& \quad d \geq \ell,
\end{align}

where the Hessian matrix \( W \in \mathbb{R}^{n \times n} \) is positive definite, \( g \in \mathbb{R}^n \) is a gradient vector, and the solution \( d \in \mathbb{R}^n \) is subject to equality constraints with matrix \( A \in \mathbb{R}^{m \times n} \) and vector \( c \in \mathbb{R}^m \) as well as lower bounds \( \ell \in \mathbb{R}^n \). We assume that \( A \) has full row rank.
The main contribution of this paper is a novel QP algorithm that exploits information already available from the solution procedure for an "initial QP,"

\[
\begin{align*}
\text{(1.2a)} & \quad \min_{d \in \mathbb{R}^n} \frac{1}{2} d^T \tilde{W} d + g_0^T d \\
\text{(1.2b)} & \quad \text{s.t. } \tilde{A} d + c_0 = 0, \\
\text{(1.2c)} & \quad d \geq \ell_0
\end{align*}
\]

in order to solve a new QP (1.1), where \( \tilde{W} \approx W \) and \( \tilde{A} \approx A \). Here, we assume that the initial QP (1.2) has been solved by a generic active-set QP solver that is capable of hot-starts (a notion made precise in section 2). Akin to the iterative refinement procedure for linear equations, iterates \( d_i \) converging to the optimal solution \( d^* \) of QP (1.1) are generated by repeatedly solving the initial QP (1.2) with different vectors \( g_i, c_i, \) and \( \ell_i \), indexed by \( i \geq 1 \),

\[
\begin{align*}
\text{(1.3a)} & \quad \min_{d \in \mathbb{R}^n} \frac{1}{2} d^T \tilde{W} d + g_i^T d \\
\text{(1.3b)} & \quad \text{s.t. } \tilde{A} d + c_i = 0, \\
\text{(1.3c)} & \quad d \geq \ell_i.
\end{align*}
\]

In order to improve convergence speed, an accelerated linear solver method, such as SQMR \[17\], is applied when the active set stops changing, and QP (1.3) then acts as a preconditioner for the linear solver method. The method is shown to converge locally if strict complementarity holds and the gradients of the active constraints are linearly independent.

The advantage of the proposed algorithm is that the factorization of the KKT matrix for QP (1.2), which is required in the active-set QP solver and involves submatrices of \( \tilde{W} \) and \( \tilde{A} \), can be reused for solving (1.1). This is beneficial in particular if multiple instances of (1.1) need to be solved. In contrast, if the active-set solver were to be applied to solve (1.1) directly, a new factorization would have to be computed from scratch for each new instance with \( W \) and \( A \) different from \( \tilde{W} \) and \( \tilde{A} \).

An additional benefit of the new QP solver is that it requires only matrix-vector products with \( W, A, \) and \( A^T \) during the solution of (1.1). This can lead to significant computational savings if the computation of the full matrix \( A \) is expensive, e.g., if the computation of the constraints (1.1b) involves the numerical solution of differential equations.

1.1. Motivation. This research is motivated by a number of applications where a successive solution of QPs with similar data is required. (In this paper, we use the term "similar" loosely to express that the vectors and matrices, as well as the corresponding solutions of the QPs, are close to each other.)

One example is nonlinear model-predictive control (NMPC), a numerical approach for optimally controlling a dynamic process (such as a chemical plant or a vehicle) in real-time. Here, at a given point \( \tau \) in time, an optimal control action is computed as the solution of a QP that is obtained from a local quadratic model of the control problem and involves some linearization of the differential equations describing the process. (See section 6.1.3 for details.) The initial conditions and exogenous system parameters are chosen according to the actual or estimated state of the system at time \( \tau \). After a small time interval \( \Delta \tau \), a new control action is computed, now using the initial conditions and system parameter values corresponding to \( \tau + \Delta \tau \). If \( \Delta \tau \) is small and the state of the system has not changed very much, the QPs solved at \( \tau \) and \( \tau + \Delta \tau \) as well as their solutions are similar.
The sequential quadratic programming (SQP) method for solving nonlinear programs (NLPs) represents another example. Consider an NLP of the form

\begin{align}
\text{(1.4a)} & \quad \min_{x \in \mathbb{R}^n} f(x) \\
\text{(1.4b)} & \quad \text{s.t. } c(x) = 0, \\
\text{(1.4c)} & \quad x \geq 0,
\end{align}

where the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and the constraint function $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $m \leq n$ are continuously differentiable. In a generic SQP method, the step $d_k$ at an iterate $x_k$ is obtained as the optimal solution of the QP (1.1) with $\ell = -x_k$, where $g = \nabla f(x_k)$ is the gradient of the objective function, $e = c(x_k)$ is the residual of the constraints, $A = \nabla c(x_k)^T$ is the Jacobian of the constraints, and $W$ is (an approximation of) the Hessian of the Lagrangian function at $x_k$ for given multiplier estimates $\lambda_k$ for the equality constraints. Here, all vectors and matrices depend on the iterate $x_k$, and, consequently, are different at each iterate of the method. However, if the iterates are close to each other, these quantities can be expected not to change very much. This is, for example, the case, when the SQP algorithm is close to convergence.

Furthermore, suppose that we are interested in solving a sequence of NLPs (1.4) that differ only slightly in $f(x)$ or $c(x)$. If the optimal solution of the new NLP is close to the optimal solution of the previous one, the SQP method might require only a small number of iterations. The corresponding QPs are often similar not only to each other, but also across the different nonlinear problems. In this setting, it may be beneficial to solve the QPs arising during the SQP algorithm with the algorithm proposed in this paper, where the QP from the last SQP iteration of the first NLP is taken as the initial QP (1.2). This can be seen as a procedure for solving a sequence of similar NLPs using hot-starts.

The solution of a sequence of closely related NLPs or QPs is also required during the execution of a branch-and-bound search for a mixed-integer nonlinear program (MINLP). Here, each node of the enumeration tree requires the solution of an NLP or QP relaxation, with different bound constraints. Moreover, during diving heuristics (see, e.g., [24, 23] and references therein) or strong-branching (see, e.g., [1, 23]), a succession of similar NLPs or QPs has to be solved.

### 1.2. Structure of the article.

This paper is organized as follows. Because our method crucially depends on the active-set method for QPs, we give a short summary in section 2 that introduces the notation and concepts necessary for the remainder of this paper. In section 3, we briefly review the iterative refinement procedure for solving a linear system of equations. Reinterpreting this technique in the context of equality-constrained optimization in section 4.1, we make the connection to the use of hot-starts of QP (1.3) for the solution of the QPs (1.1). This is then generalized in section 4.2 to handle inequality constraints. An accelerated version is presented in section 5, where the solution of QP (1.3) is used as a preconditioner within an iterative linear solver method. In section 6.1, we explore the performance of the new QP solver in the context of an optimal control application. Section 6.2 examines the performance of the new method within an SQP framework applied to sequences of randomly generated NLPs with perturbed data. We conclude with some final remarks in section 7.

**Notation.** Given a vector (or vector-valued function) $x \in \mathbb{R}^n$, we denote by $x^{(i)}$ the $i$th component of this vector. Given a set $\mathcal{S} \subseteq \{1, \ldots, n\}$, we denote by $x^\mathcal{S}$ the vector composed from elements $x^{(i)}$ with indices $i \in \mathcal{S}$, and $\mathcal{S}^C$ denotes the complement.
of $S$ in $\{1, \ldots, n\}$. To simplify the notation, we write $(x,y)$ for the concatenation $(x^T, y^T)^T$ of two vectors $x$ and $y$. With $\| \cdot \|$, we refer to any vector norm (and its corresponding matrix norm) for which $\|x\| \leq \|(x,y)\|$. The active set at a feasible point $d$ of QP (1.1) is denoted by $A(d) = \{j : d^{(j)} = \ell^{(j)}\}$, and its complement is $F(d) = A(d)^C$. Similarly, $A_i(d) = \{j : d^{(j)} = \ell^{(j)}\}$ with the complement denoted by $F_i(d)$ for the initial QP (1.2), $i = 0$, and the QPs (1.3), $i > 0$.

2. Active-set solvers. To solve QP (1.1), the proposed algorithm utilizes repeated solutions of QP (1.3) where the matrices $W$ and $\tilde{A}$ remain constant, and only the vectors $g_i$, $c_i$, and $l_i$ change. Important details of the QP algorithm proposed in this article rely on the use of an active-set solver for this repeated solution. One key observation is that an active-set QP solver can often compute subsequent solutions with identical matrices $W$ and $\tilde{A}$ much faster than in the case when the matrices change. Hence, in this section we briefly discuss the linear algebra tasks for the optimality system of QP (1.3) carried out by a typical active-set QP solver.

An active-set QP solver for QP (1.3) maintains a guess $A$ of the set $A(d_*)$ of active variable bounds (1.3c) that are active at the optimal solution $d_*$. (To simplify the notation, we drop the subscript $i$ in $A_i$ for the remainder of this section.) An iterate of the QP solver is computed from the solution of the linear system

\begin{equation}
\begin{bmatrix}
\tilde{W}^{FF} & \tilde{W}^{FA} \\
\tilde{W}^{AF} & \tilde{W}^{AA}
\end{bmatrix}
\begin{bmatrix}
(A^F)^T \\
(A^A)^T
\end{bmatrix}
= -
\begin{bmatrix}
g_i^F \\
g_i^A
\end{bmatrix}
\end{equation}

(2.1)

Here, $\tilde{W}^{FF}$ denotes the submatrix of $\tilde{W}$ with rows corresponding to $F = A^C$ and columns corresponding to $A$, $A^F$ is the submatrix of $A$ with the columns corresponding to $F$, and $\tilde{W}^{FF}$, $\tilde{W}^{AF}$, $\tilde{W}^{AA}$, and $\tilde{A}^A$ are defined in a similar manner. If $d^F \geq l_i^F$ and $\mu^A \geq 0$, the current iterate is optimal. Otherwise, $A$ is updated, usually by adding or removing one variable.

Observing that $d^A = l_i^A$, the above linear system can be reduced to

\begin{equation}
\begin{bmatrix}
\tilde{W}^{FF} & (A^F)^T \\
\tilde{A}^F & 0
\end{bmatrix}
\begin{bmatrix}
(d^F) \\
(\lambda)
\end{bmatrix}
= -
\begin{bmatrix}
g_i^F + \tilde{W}^{FA} l_i^A \\
(\lambda)
\end{bmatrix}
\end{equation}

(2.2)

and the multipliers corresponding to the bound constraints are computed from the second block equation in (2.1),

\begin{equation}
\mu^A = g_i^A + \tilde{W}^{FA} d^F + \tilde{W}^{AA} l_i^A + (\tilde{A}^A)^T \lambda.
\end{equation}

(3.3)

Therefore, during each iteration of the active-set QP algorithm, a linear system of the form (2.2) has to be solved. Different methods use different techniques to solve this linear system (e.g., null space methods [14, 15], Schur complement methods [3, 18, 19]), all of which involve the factorization of matrices constructed from $A^F$ and $\tilde{W}^{FF}$. To avoid large computational costs, the factorization is not computed from scratch in each iteration of the QP solver; instead, since typically only one element enters or leaves the active set $A$, the factorization is updated in an efficient manner.

In this article, we say that an active-set QP solver performs a hot-start if it uses the optimal active set $A$ from a previously solved QP as the starting guess for a new QP and if the internal factorization corresponding to the optimal solution of the
previous QP is reused. The latter can only be done if the matrices, \( \tilde{A} \) and \( \tilde{W} \) in (2.2), remain the same. In this case, if the new optimal active set is similar to the one from the previous QP (e.g., because \( g_i, c_i, \) and \( l_i \) have not changed much), only a few iterations of the QP solver are required, and the solution for the new QP can be obtained very quickly. If all of the vectors \( g_i, c_i, \) and \( l_i \) change, i.e., both primal and dual feasibility are destroyed, a parametric QP solver \([4, 12]\) is a suitable choice since it does not require a primal or dual feasible starting point.

A practical QP solver has to be able to handle degeneracy, i.e., situations in which \( \tilde{A}^F \) does not have full row rank. For this purpose, instead of constructing (2.1) with the set \( A = \{ j : d = t_i^{(j)} \} \), where \( d \) is the current QP solver iterate, it is common to use a working set \( W \) chosen as a maximal subset of \( A \) so that \( \tilde{A}(W^c) \) has full row rank. For simplicity, however, we largely assume in this paper that \( \tilde{A}^F \) has full row rank.

3. Iterative refinement for linear systems. In this section, we review the iterative refinement method for linear systems before addressing the solution of optimization problems in section 4.1.

Suppose we are interested in the solution of a system of linear equations

\[
Mx = b, \tag{3.1}
\]

where \( M \) is a nonsingular matrix. We assume that a factorization of a nonsingular matrix \( \tilde{M} \) that is not too different from \( M \) is available, and that operations with \( \tilde{M}^{-1} \) can hence be carried out. After initializing \( x_1 = 0 \), we repeat for \( i = 1, 2, 3, \ldots \)

\[
\begin{align*}
    p_i &= -\tilde{M}^{-1}(Mx_i - b), \tag{3.2a} \\
    x_{i+1} &= x_i + p_i, \tag{3.2b}
\end{align*}
\]

until \( x_{i+1} \) is deemed to be a sufficiently good solution of (3.1). After eliminating \( p_i \) and rearranging terms, we see that the recurrence (3.2) satisfies the fixed-point iteration

\[
x_{i+1} = \left( I - \tilde{M}^{-1}M \right) x_i + \tilde{M}^{-1}b. \tag{3.3}
\]

This can also be written as

\[
x_{i+1} - x_* = \left( I - \tilde{M}^{-1}M \right) (x_i - x_*), \tag{3.4}
\]

where \( x_* \) solves (3.1). Therefore, the sequence of iterates \( x_i \) converges to \( x_* \) if

\[
\left\| I - \tilde{M}^{-1}M \right\| < 1 \tag{3.5}
\]

for some norm \( \| \cdot \| \). For later reference, we note that we can rewrite (3.3) as

\[
\tilde{M}x_{i+1} = \left( \tilde{M} - M \right) x_i + b. \tag{3.6}
\]

4. Hot-started active-set solvers for quadratic programming. In this section, we transfer the idea of iterative refinement for linear systems first to equality-constrained QPs and then to inequality-constrained QPs.
4.1. Equality-constrained problems. The iterative refinement scheme can be applied to QPs with only equality constraints, i.e., problem (1.1), where (1.1c) is absent. In this case, the first-order optimality conditions of the QP can be stated as

\[
\begin{bmatrix}
W & A^T \\
A & 0
\end{bmatrix}
\begin{bmatrix}
d_i \\
x
\end{bmatrix}
= -
\begin{bmatrix}
g \\
c
\end{bmatrix}.
\]

Here, and in the remainder of this paper, we assume that \(W\) is positive definite, so that the solution of (4.1) is a global solution of the QP. Choosing \(M, x,\) and \(b\) as indicated in (4.1), the iterative refinement procedure (3.2) applied to (4.1) becomes

\[
\begin{align*}
d_{i+1} &= d_i + p_i, \\
\lambda_{i+1} &= \lambda_i + p_i^A
\end{align*}
\]

with refinement iterates indexed by \(i\), where the steps satisfy

\[
\begin{bmatrix}
\tilde{W} & \tilde{A}^T \\
\tilde{A} & 0
\end{bmatrix}
\begin{bmatrix}
p_i \\
p_i^A
\end{bmatrix}
= -
\begin{bmatrix}
g \\
c
\end{bmatrix} - \begin{bmatrix}(Wd_i + A^T\lambda_i) \\
(W - \tilde{W})d_i + (A - \tilde{A})^T\lambda_i
\end{bmatrix}
\]

(cf. (3.6)). The linear system (4.3) states the first-order optimality conditions for the QP

\[
\begin{align*}
\min_p & \quad \frac{1}{2}p^T\tilde{W}p + (g + Wd_i + A^T\lambda_i)^T p \\
\text{s.t.} & \quad \tilde{A}p + (c + Ad_i) = 0
\end{align*}
\]

Therefore, \(p_i\) can be obtained equivalently as the optimal solution of this QP, and \(p_i^A\) are the optimal multipliers for (4.4b). In summary, the iterative refinement procedure consists of generating iterates using the update (4.2), where the steps are computed as the solution of the QP (4.4).

We stress that the matrices in (4.4), i.e., \(\tilde{W}\) and \(\tilde{A}\), remain unchanged over the refinement iterations \(i\), and only the vectors in the objective gradient and constraint right-hand side of this QP vary. Therefore, a QP solver capable of hot-starts will often be able to compute solutions for each QP (4.4) very rapidly, once an initial QP (1.2) has been solved (see section 2). Note that for setting up (4.4), the matrices \(W\) and \(A\) are needed, but they are used in matrix-vector products only.

In order to obtain a geometric interpretation of QP (4.4), we note that, analogously to (3.6), (4.3) and (4.2) can be rearranged to give

\[
\begin{bmatrix}
\tilde{W} & \tilde{A}^T \\
\tilde{A} & 0
\end{bmatrix}
\begin{bmatrix}
d_{i+1} \\
\lambda_{i+1}
\end{bmatrix}
= -
\begin{bmatrix}
g \\
c
\end{bmatrix} - \begin{bmatrix}(W - \tilde{W})d_i + (A - \tilde{A})^T\lambda_i \\
(A - \tilde{A})d_i
\end{bmatrix}
\]

or, expressed as a QP,

\[
\begin{align*}
\min_d & \quad \frac{1}{2}d^T\tilde{W}d + (g + (W - \tilde{W})d_i + (A - \tilde{A})^T\lambda_i)^T d \\
\text{s.t.} & \quad \tilde{A}d + (c + (A - \tilde{A})d_i) = 0
\end{align*}
\]

where the optimal multipliers for (4.6b) are the new iterates \(\lambda_{i+1}\). Here, the objective gradient and constraint right-hand side are modified to compensate for the difference in the matrices.
Algorithm 1. Solving QP (1.1) using hot-starts for QP (4.7) based on iterative refinement.

1: Given: Initial iterate \((d_1, \lambda_1, \mu_1)\) with \(d_1 \geq \ell\).
2: for \(i = 1, 2, 3, \ldots\) do
3:   Solve QP (4.7).
4:   Let \(p_i\) be the optimal solution of QP (4.7) and set \(d_{i+1} = d_i + p_i\).
5:   Let \(p_{\lambda i}\) be the optimal multipliers for (4.7b) and set \(\lambda_{i+1} = \lambda_i + p_{\lambda i}\).
6:   Let \(\mu_{i+1}\) be the optimal multipliers for (4.7c).
7: end for

This correction is depicted in Figure 1. The lines and contours represent the constraints and objective; those of the original problem (1.1) are indicated by dashes, and the solid lines represent (4.6). In the left plot, which corresponds to the first refinement iteration \(i = 1\), the solution \(d_1\) for (4.6) is away from the desired solution \(d^*\). As the iterative refinement progresses, the terms involving \(W - \tilde{W}\) and \(A - \tilde{A}\) act as corrections for the positions of the objective and constraints of the QP (4.6), so that, as demonstrated in the right plot, the iterates \(d_i\) approach \(d^*\).

4.2. Inequality-constrained problems. To handle QPs (1.1) with inequality constraints, we augment QP (4.4) with bound constraints that ensure that the refinement iterates (4.2a) always satisfy the original bound constraints (1.1c),

\[
\begin{align*}
(4.7a) & \quad \min_p \frac{1}{2} p^T \tilde{W} p + (g + W d_i + A^T \lambda_i)^T p \\
(4.7b) & \quad \text{s.t. } \tilde{A} p + (c + A d_i) = 0, \\
(4.7c) & \quad d_i + p \geq \ell.
\end{align*}
\]

The resulting method is formally stated as Algorithm 1.

The iterates \((d_i, \lambda_i, \mu_i)\) now include the bound multipliers \(\mu_i\) corresponding to (1.1c). The optimal solution of (4.7) provides the step \(p_i\), the optimal multipliers for (4.7b) provide the step \(p_{\lambda i}\), and the iterates \(d_{i+1}\) and \(\lambda_{i+1}\) are updated according to (4.2).

The iterates for the bound multipliers \(\mu_{i+1}\) are updated in a different manner; they are simply set to the optimal multipliers corresponding to (4.7c) and, in contrast to \(\lambda_i\), do not appear in the QP gradient (4.7a). In fact, the updates for the \(d_i\) and \(\lambda_i\) iterates can be performed without the knowledge of the \(\mu_i\) iterates, and the algorithm can be executed without explicitly tracking \(\mu_i\). To see that this is reasonable, suppose for the moment that the active set \(\mathcal{A}_i = \mathcal{A}(d_i)\) at the optimal solution of the original QP (1.1) is known and that we apply the procedure in section 4.1 directly to the equality.

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constrained QP with the original equality constraints and active bound constraints,

\begin{align}
(4.8a) \quad & \min_{d \in \mathbb{R}^n} \frac{1}{2} d^T W d + g^T d \\
(4.8b) \quad & \text{s.t. } Ad + c = 0, \\
(4.8c) \quad & d^A_* = \ell^A_*. 
\end{align}

Then, the multiplier iterates \( \lambda_i \) in (4.2b) consist of the multipliers \( \lambda_{\text{orig}} \) for the original
constraints (1.1b), and the multipliers \( \mu^A_* \) for the active bound constraints (1.1c), i.e., \( \lambda_i = (\lambda_{\text{orig}}^i, \mu^A_*^i) \). For the extended QP (4.8a), the term involving \( \lambda_i \) in the right-hand side of (4.5) then becomes

\[
\left( \begin{bmatrix} A \\ I \end{bmatrix} - \begin{bmatrix} \tilde{A} \\ I \end{bmatrix} \right) \begin{bmatrix} \lambda_{\text{orig}}^i \\ \mu_i^A_* \end{bmatrix} = (A - \tilde{A})^T \lambda_{\text{orig}}^i.
\]

Therefore, the bound multipliers \( \mu_i^A_* \) are not needed in order to compute the new iterate. Consequently, if the active set of (4.7) has settled to the optimal active set \( \mathcal{A}_* \), it is not necessary to track the bound multipliers explicitly in order to execute the iterative refinement algorithm proposed in the previous section. Our algorithm uses this update strategy also when the active set of (4.7) may not have settled yet.

In a practical setting, a suitable termination criterion is required. In this paper, we use the function

\[
\Phi(d, \lambda) = \left\| \begin{bmatrix} \min\{d - l, g + Wd + A^T \lambda\} \\ Ad + c \end{bmatrix} \right\|_2.
\]

It is easy to verify that \( \Phi(d_*, \lambda_*) = 0 \) if and only if \( (d_*, \lambda_*) \) is optimal. Note that an explicit knowledge of the bound multipliers \( \mu \) is again not required. We may also use \( \Phi(d, \lambda) \) as a means to monitor whether Algorithm 1 is diverging or cycling.

Before proving the main theorem of this section, we provide a lemma discussing local properties of the iterates under some regularity assumptions. We require the notion of strict complementarity.

**Definition 4.1.** Let \( (d_*, \lambda_*) \) be the unique optimal primal-dual solution of QP (1.1) with active set \( \mathcal{A}(d_*) \). We say that the strict complementarity condition holds if \( \lambda_j^0 > 0 \) for all \( j \in \mathcal{A}(d_*) \).

**Lemma 4.2.** Suppose that \( \tilde{W} \) and \( W \) are positive definite and that QP (4.7) is feasible for each \( i \) (so that Algorithm 1 is well-defined). Further assume that \( (d_*, \lambda_*, \mu_*) \) is the unique optimal primal-dual solution of QP (1.1) with active set \( \mathcal{A}_* = \mathcal{A}(d_*) \), that strict complementarity holds, and that \( \tilde{A}^F_* \) and \( A^F_* \) with \( F_* = \mathcal{A}_C^C \) have full row rank.

Then there exists \( \epsilon > 0 \) and \( c_1 > 0 \) so that for all \( i \) with \( (d_i, \lambda_i) \in B_\epsilon := \{(d, \lambda) \mid \| (d, \lambda) - (d_*, \lambda_*) \| \leq \epsilon \} \), we have \( \mathcal{A}(d_{i+1}) = \mathcal{A}_* \) and \( \| (d_{i+1}, \lambda_{i+1}) - (d_*, \lambda_*) \| \leq c_1 \| (d_i, \lambda_i) - (d_*, \lambda_*) \| \) if \( (d_{i+1}, \lambda_{i+1}) \) is obtained from the updates in steps 4 and 5 of Algorithm 1.

**Proof.** First note that \( (d_*, \lambda_*, \mu_*) \) satisfies the KKT conditions for (1.1), i.e.,

\begin{align}
(4.10a) \quad & g + Wd_* + A^T \lambda_* - \mu_* = 0, \\
(4.10b) \quad & c + Ad_* = 0, \\
(4.10c) \quad & \mu_* \geq 0, \\
(4.10d) \quad & d_* \geq \ell, \\
(4.10e) \quad & \mu_*^T (d_* - \ell) = 0.
\end{align}
Considering QP (4.7) and following arguments similar to the derivation of (4.6), one can show that \( d_{i+1} \) is the solution of

\[
\begin{align*}
(4.11a) & \quad \min_d \frac{1}{2}d^T \tilde{W}d + (g + (W - \tilde{W})d_i + (A - \tilde{A})^T \lambda_i)^T d \\
(4.11b) & \quad \text{s.t. } \tilde{A}d + (c + (A - \tilde{A})d_i) = 0, \\
(4.11c) & \quad d \geq \ell,
\end{align*}
\]

and \( (\lambda_{i+1}, \mu_{i+1}) \) are the corresponding optimal multipliers. The KKT conditions for this QP are given as

\[
\begin{align*}
(4.12a) & \quad \tilde{W}d_{i+1} + g + (W - \tilde{W})d_i + (A - \tilde{A})^T \lambda_i + \tilde{A}^T \lambda_{i+1} - \mu_{i+1} = 0, \\
(4.12b) & \quad \tilde{A}d_{i+1} + c + (A - \tilde{A})d_i = 0, \\
(4.12c) & \quad \mu_{i+1} \geq 0, \\
(4.12d) & \quad d_{i+1} \geq \ell, \\
(4.12e) & \quad (\mu_{i+1})^T (d_{i+1} - \ell) = 0.
\end{align*}
\]

Suppose for the moment that \( d_i = d_* \) and \( \lambda_i = \lambda_* \) and substitute (4.10a) and (4.10b) into (4.12a) and (4.12b). Rearranging terms, one can verify that then \( (d_{i+1}, \lambda_{i+1}, \mu_{i+1}) = (d_*, \lambda_*, \mu_*) \) satisfies the KKT conditions (4.12) and is therefore the unique optimal solution of (4.11). Since (4.11c) and (1.1c) are identical, the active set for (4.11) is \( A_* \).

Because of the full row rank of \( \tilde{A}^T \), the linear independence constraint qualification (LICQ) holds at the solution of (4.11) if \( (d_i, \lambda_i) = (d_*, \lambda_*) \). Since strict complementarity also holds and \( \tilde{W} \) is positive definite by assumption, standard sensitivity results hold (see, e.g., [13]), and there exists \( \epsilon > 0 \) so that for \( (d_i, \lambda_i) \in B_\epsilon \) (i.e., the gradient in (4.11a) and the constant term in (4.11b) vary sufficiently little), the active set of (4.11) does not change and is identical to that of QP (1.1). Hence, \( A(d_{i+1}) = A_* \).

Let \( (d_i, \lambda_i) \in B_\epsilon \). Noting that \( \mu_{i+1} = 0 \) and \( d_{i+1} = d_* \), substituting again (4.10a) and (4.10b) into (4.12a) and (4.12b), and rearranging terms, it can be seen that

\[
\begin{bmatrix}
\tilde{W}^*, \tilde{W}^* A_* \\
\tilde{W}^* A_* , \tilde{W}^* A_* (\tilde{A}^T)^T - I \\
\tilde{A}^T, \tilde{A}^T A_* \\
0, -I
\end{bmatrix}
\begin{bmatrix}
(\lambda_{i+1} - \lambda_*) A_* \\
\mu_{i+1} - \mu_* A_* \\
d_{i+1} - d_* \\
(g_{i}^* - g_{i})^T (A - \tilde{A}) (d_i - d_*)
\end{bmatrix} = 0,
\]

where \( g_i = (W - \tilde{W})(d_i - d_*) + (A - \tilde{A})^T (\lambda_i - \lambda_*) \). Since \( \tilde{W} \) is positive definite and \( \tilde{A}^T \) has full row rank, the matrix \( K_A \) in this linear system is nonsingular; see, e.g., [25, Lemma 16.1]. The claim then follows with \( c_1 = ||I - K_A^{-1} K_A|| \), where \( K_A \) is the matrix in (4.13) with \( \tilde{W} \) and \( \tilde{A} \) replaced by \( W \) and \( A \), respectively. □
Lemma 4.2 characterizes the local behavior of the iterates only. It is not a
contraction result unless \( c_1 < 1 \) also holds, which was not required to prove the
lemma. It is the purpose of the next theorem to show that Algorithm 1 indeed
has desirable convergence properties. In particular, the method cannot converge to
spurious solutions, and local convergence is guaranteed under regularity assumptions
if the matrix data satisfies a contraction condition.

**Theorem 4.3.** Suppose that \( \tilde{W} \) and \( W \) are positive definite and that QP (4.7) is
feasible for each \( i \). Then the following statements hold true for the sequence \((d_i, \lambda_i, \mu_i)\)
generated by Algorithm 1:

(i) If \((d_i, \lambda_i)\) converges to \((d_\ast, \lambda_\ast)\), then \(d_\ast\) is the unique optimal solution of
QP (1.1), and \(\lambda_\ast\) are optimal multipliers.

(ii) Suppose that the assumptions of Lemma 4.2 hold and that in addition

\[
(4.14) \quad c_2 := \left\| I - \left[ \begin{matrix} W_{\ast \ast} & (\tilde{A}_{\ast \ast})^T \\ \tilde{A}_{\ast \ast} & 0 \end{matrix} \right]^{-1} \left[ \begin{matrix} W_{\ast \ast} & (\tilde{A}_{\ast \ast})^T \\ \tilde{A}_{\ast \ast} & 0 \end{matrix} \right] \right\| < 1.
\]

Then, if \((d_i, \lambda_i)\) is sufficiently close to \((d_\ast, \lambda_\ast)\), the sequence generated by
Algorithm 1 converges to \((d_\ast, \lambda_\ast, \mu_\ast)\).

**Proof.** (i) Recall that the iterates satisfy the optimality condition (4.12) of the
QP (4.11). Because \((d_i, \lambda_i)\) converges to \((d_\ast, \lambda_\ast)\), we have that \(\mu_i\) is bounded.
Consequently, there is a subsequence \(\mu_{i_j}\) that converges to some \(\mu_\ast\). Taking the limit in
(4.12) as \(i_j \to \infty\), we see that the limit point satisfies the optimality conditions (4.10).
Hence, \(d_\ast\) is an optimal solution of QP (1.1) with optimal multipliers \(\lambda_\ast\) and \(\mu_\ast\). The
uniqueness of \(d_\ast\) follows from the positive-definiteness of \(W\).

(ii) Let \(\epsilon > 0\) and \(c_1 > 0\) be the constants from Lemma 4.2. Set \(\tilde{\epsilon} = \epsilon/\max\{1, c_1\}\)
and let \((d_i, \lambda_i)\) \(\in B_{\tilde{\epsilon}} \subseteq B_\epsilon\). Then \(A(d_{i+1}) = A_\ast\) and \((d_{i+1}, \lambda_{i+1}) \in B_{\tilde{\epsilon}} \subseteq B_\epsilon\) by
Lemma 4.2. Applying this argument a second time, it follows that \(A(d_{i+2}) = A_\ast\).

Therefore, (4.13) holds with \(i\) replaced by \(i + 1\), and \(d_{i+2} = d_{i+1}^\ast = d_\ast^\ast = l^\ast\).
Because then \(d_{i+2}^{A_i} - d_{i+1}^{A_i} = d_{i+1}^{A_i} - d_\ast^A_i = 0\), (4.13) can be reduced to

\[
\left[ \begin{matrix} W_{\ast \ast} & (\tilde{A}_{\ast \ast})^T \\ \tilde{A}_{\ast \ast} & 0 \end{matrix} \right] \left( d_{i+2}^\ast - d_\ast^\ast \right) \left( \lambda_{i+2} - \lambda_\ast \right) = - \left( \left[ \begin{matrix} W_{\ast \ast} & (\tilde{A}_{\ast \ast})^T \\ \tilde{A}_{\ast \ast} & 0 \end{matrix} \right] - \left[ \begin{matrix} W_{\ast \ast} & (\tilde{A}_{\ast \ast})^T \\ \tilde{A}_{\ast \ast} & 0 \end{matrix} \right] \right) \left( d_{i+1}^{A_i} - d_\ast^{A_i} \right).
\]

Using (4.14), we obtain

\[
(4.15) \quad \|(d_{j+1}, \lambda_{j+1}) - (d_\ast, \lambda_\ast)\| \leq c_2 \|(d_j, \lambda_j) - (d_\ast, \lambda_\ast)\|
\]

with \(j = i + 1\). From the assumption \(c_2 < 1\), it follows that the new iterate \((d_{i+2}, \lambda_{i+2})\)
also lies in \(B_\epsilon\). Repeating this argument, we see that \((d_j, \lambda_j) \in B_\epsilon\) and (4.15) holds
for all \(j \geq i + 1\), and we obtain \((d_j, \lambda_j) \to (d_\ast, \lambda_\ast)\).

5. **Acceleration by preconditioned linear solver.** In this section, we discuss
how we can further utilize hot-starts for QP (1.3), this time using an iterative linear
solver for the linear system (4.3) that aims at obtaining a better convergence speed than the linear rate of the simple iterative refinement scheme (3.2). The crucial observation is that we can exploit the existing factorization in the active-set QP solver for (1.3) as a preconditioner for the iterative linear solver at very little cost.

Let us assume for the moment that the optimal active set \( A^\ast \) is known. In that case, we seek the solution of the linear system

\[
\begin{bmatrix}
W^{FF} \\
A^F
\end{bmatrix} 
\begin{bmatrix}
(A^F)^T \\
0
\end{bmatrix} 
\begin{bmatrix}
d^F \\
\lambda
\end{bmatrix} = - \begin{bmatrix}
g + W^{FA} d^A \\
c + A^A d^A
\end{bmatrix}
\]

with \( A = A^\ast, \quad F = A^C \), and we set \( d^A = \ell^A \) (see also (2.2)). The disadvantage of the fixed-point iteration (3.2), which is the basis of the algorithm proposed so far, lies in its potentially slow linear convergence rate and the requirement of a contraction condition; cf. (3.5). To speed up the convergence, we may use some accelerated iterative linear solver method, such as GMRES [28], LSQR [26], LSMR [16], or SQMR [17]. Note that these methods converge independently of some contraction condition. For good practical performance, however, they require the application of an appropriate preconditioner. We propose using a linear solver with a preconditioner constructed from \( M \) by replacing \( W \) and \( A \) with \( \tilde{W} \) and \( \tilde{A} \), that is, the application of the preconditioner requires the solution of

\[
\begin{bmatrix}
\tilde{W}^{FF} \\
\tilde{A}^F
\end{bmatrix} 
\begin{bmatrix}
(A^F)^T \\
0
\end{bmatrix} 
\begin{bmatrix}
\tilde{d}^F \\
\lambda_j
\end{bmatrix} = \begin{bmatrix}
r_j \\
s_j
\end{bmatrix}
\]

during iteration \( j \) of the iterative linear solver. In our context, the precise definition of the right-hand side \( r_j \) and \( s_j \) is not important; these vectors are defined by the specific iterative linear solver method. The solution \( (\tilde{d}^F_j, \lambda_j) \) provides preconditioned quantities required by the particular linear solver. As before, we make the key observation that the solution of (5.2) can be obtained equivalently by solving the QP

\[
\begin{align*}
&\min_z \frac{1}{2} z^T \tilde{W} z - r_j^T z^F \\
&\quad \text{s.t. } \tilde{A} z - s_j = 0, \\
&\quad z^A \geq 0.
\end{align*}
\]

However, the optimal active set \( A^\ast \) is not known in advance. In our algorithm, we start the iterative linear solver for (5.1) if the active set no longer changes in Algorithm 1 because we might have found the optimal active set. Still, it is important to detect nonoptimal active sets. For this purpose, we obtain the preconditioned quantities \( z^F \) and \( z^\lambda \) from the modified QP

\[
\begin{align*}
&\min_z \frac{1}{2} z^T \tilde{W} z - r_j^T z^F + (g^A + W^{FA} \tilde{d}^F_j + W^{AA} d^A + (A^A)^T \tilde{\lambda}_j)^T z^A \\
&\quad \text{s.t. } \tilde{A} z - s_j = 0, \\
&\quad z^A \geq 0,
\end{align*}
\]

where \( \tilde{d}^F_j \) and \( \tilde{\lambda}_j \) are the iterates of the linear solver. This QP differs from (5.3) in
the nonzero gradient for the active variables and in the relaxation of the equality constraints (5.3c).

Our rationale for this modification is as follows. The multipliers of the active variables $d^A_c = \ell^A_c$ in the QP corresponding to (5.1) are given by (2.3) with $(d^F, \lambda)$ substituted by the iterates $(\bar{d}^F_j, \bar{\lambda}_j)$ of the iterative linear solver. If the set $F$ is optimal, the objective gradient term in QP (5.4) for $z^A$ converges to the optimal multipliers for the active bound constraints and is therefore nonnegative. If $F$ is not optimal, some component in this term will become negative, and then, as we will see in the proof of Theorem 5.1, the optimal $z^A$ will no longer be zero at some point. Hence, if we observe $z^A > 0$, we take this as an indication that too many variables have been considered active. Note that QP (5.4) can be solved using a hot-start for QP (1.3).

We point out that in principle, any preconditioner could be applied for the iterative solution of (5.1), but its construction typically requires extra work. In contrast, our preconditioner is readily available at little additional cost, because its application requires only an internal solve in the active-set QP solver, using the existing factorization of the matrix in (5.2).

Algorithm 2 details the overall method and makes use of an iterative refinement loop indexed by $i$ and an (inner) iterative solver loop indexed by $j$. In the former, the algorithm applies the iterative refinement steps (steps 4 and 25) in the same way as in Algorithm 1. Recall that the iterates $\mu^A$ for the bound multipliers are not required in order to execute the algorithm and are omitted here for simplicity. During this procedure, the method keeps track of the active set $A_i$. If the active set is identical in two consecutive iterations and the most recent step was an iterative refinement step (i.e., $\text{ref}_i$ flag = true), the algorithm starts the accelerated iterative linear solver loop (steps 10–23) from the current refinement iterate in order to solve (5.1) for the current active set. Here, applications of the preconditioner are performed by solving the QP (5.4). The iterative linear solver method is interrupted if either $z^A \neq 0$ (indicating that the active set might be too large) or if the iterative linear solver iterate violates the (ignored) bounds on the free variables (indicating that the active set might be too small). In either case, the algorithm reverts to the iterative refinement loop from the most recent feasible iterate of the iterative linear solver method. Note that the details of the iterative linear solver computations in steps 11 and 16 are left vague, since they depend on the particular method. For concreteness, Appendix A provides an explicit version of Algorithm 2 for the SQMR linear solver method.

The flag $\text{ref}_i$ flag is included for two reasons: First, it ensures that at least one iterative refinement step is taken between two executions of the iterative linear solver loop. In this way, the initial iterate in the second execution is different from the final iterate in the first execution. This prevents cycling in case the iterative linear solver is interrupted in its first iteration without taking any step.

Second, if $A_i = A_{i-1}$ and the most recent iteration was an iterative refinement step (step 25), it is guaranteed that $p^A_{i-1} = 0$. This is desirable if $p_i$ from (4.7) happens to be identical to the solution $z$ of the preconditioning QP (5.4) in the first iteration of the chosen iterative linear solver methods. This is the case for SQMR, and therefore a renewed solution of the iterative refinement QP in step 12 of the first SQMR iteration can then be skipped.

We note that if the iterative linear solver is interrupted before taking a step, we have $(d_{i+1}, \lambda_{i+1}) = (d_i, \lambda_i)$ and the iterative refinement QP solution $(p_i, p^A_i)$ can be used in the next iteration $i + 1$ without solving QP (4.7) again.
Algorithm 2. (1QP) Solving QP (1.1) using hot-starts for QPs (4.7) and (5.4), accelerated version.

1: Given: Initial iterates $d_i \geq \ell$ and $\lambda_1$.
2: Initialize: $A_0 = \{ l \mid d_i^{(l)} = \ell^{(l)} \}$ and $\text{ref\_flag} \leftarrow \text{false}$.
3: for $i = 1, 2, 3, \ldots$ do
4: \quad Solve QP (4.7) to obtain optimal solution $p_i$ with optimal multipliers $p_i^\lambda$.
5: \quad Determine the active set $A_i = \{ l \mid d_i^{(l)} + p_i^{(l)} = \ell^{(l)} \}$.
6: \quad if $A_i = A_{i-1}$ and $\text{ref\_flag} = \text{true}$ then
7: \quad \quad Set $\text{ref\_flag} \leftarrow \text{false}$.
8: \quad \quad Fix active set $A \leftarrow A_i$ and $F \leftarrow A^C$.
9: \quad \quad Initialize iterative linear solver for solving (5.1) with iterate $(\bar{d}_i^F, \bar{\lambda}_1) \leftarrow (d_i^F, \lambda_1)$.
10: \quad for $j = 1, 2, 3, \ldots$ do
11: \quad \quad Perform iterative linear solver computations up to application of preconditioner.
12: \quad \quad Apply preconditioner by solving QP (5.4).
13: \quad \quad if $z_j^A \neq 0$ in optimal solution of QP (5.4) then
14: \quad \quad \quad Update refinement iterate $(d_{i+1}^F, d_{i+1}^A, \lambda_{i+1}) \leftarrow (d_j^F, \ell^A, \bar{\lambda}_j)$; break.
15: \quad \quad end if
16: \quad \quad Continue to perform iterative linear solver computations to obtain $(d_{j+1}^F, \bar{\lambda}_{j+1})$.
17: \quad \quad if $d_{j+1}^F \geq \ell^F$ then
18: \quad \quad \quad Update refinement iterate $(d_{i+1}^F, d_{i+1}^A, \lambda_{i+1}) \leftarrow (d_j^F, \ell^A, \bar{\lambda}_j)$; break.
19: \quad \quad end if
20: \quad \quad if $(d_{j+1}^F, \bar{\lambda}_{j+1})$ solves (5.1) then
21: \quad \quad \quad Return optimal solution $(d_*, \lambda_*)$ with $d_*^F = d_{j+1}^F$, $d_*^A = \ell^A$, and $\lambda_* = \bar{\lambda}_{j+1}$.
22: \quad \quad end if
23: \quad end for
24: else
25: \quad Update $d_{i+1} = d_i + p_i$ and $\lambda_{i+1} = \lambda_i + p_i^\lambda$.
26: \quad Set $\text{ref\_flag} \leftarrow \text{true}$.
27: end if
28: if $(d_{i+1}, \lambda_{i+1})$ solves (1.1) then
29: \quad Return optimal solution $(d_*, \lambda_*) = (d_{i+1}, \lambda_{i+1})$.
30: end if
31: end for

Before stating the convergence properties of this method, we make the following assumptions on the iterative linear solver.

Assumptions 1. Assume that the iterative linear solver in the inner loop of Algorithm 2 (steps 9–12, 16, and 23) has the following properties:

(i) If the iterates generated by the linear solver method converge and the matrix in (5.2) is nonsingular, the limit point of the iterates satisfies the linear system (5.1) and the quantities $z_j^F$, $z_j^A$, $r_j$, and $s_j$ in the preconditioning system (5.2) converge to zero.

(ii) If the matrices in (5.1) and (5.2) are nonsingular, the iterates generated by the linear solver converge to the solution of (5.1) from any starting point.
(iii) If the matrices in (5.1) and (5.2) are nonsingular, there exists a constant $c_3 > 0$ so that
\begin{equation}
\max \left\{ \|z^F\|, \|z^\ell\|, \|r_j\|, \|s_j\|, \left\| \frac{d^F_j}{\lambda_j} \right\|, \left\| \frac{d^\ell_j}{\lambda_j} \right\| \right\} \leq c_3 \left\| \frac{d^F_1}{\lambda_1} \right\| - \left\| \frac{d^\ell_1}{\lambda_1} \right\|
\end{equation}
for all $d^F_j$ and $\lambda_1$, and for all $j$, where $(d^F_1, \lambda_1)$ is the solution of (5.1).

We point out that the LSQR [26] and LSMR [16] solvers satisfy these assumptions, as proved in [16].

The following theorem shows that Algorithm 2 cannot converge to spurious solutions and that local convergence is guaranteed under certain regularity assumptions.

**Theorem 5.1.** Suppose that Assumption 1 hold, that $W$ and $W$ are positive definite, that $QP$ (4.7) and $QP$ (5.4) are always feasible (so that Algorithm 2 is well-defined), and that the matrix in (5.2) is nonsingular whenever the preconditioning $QP$ (5.4) is solved in Step 12. Furthermore, assume that the algorithm does not terminate at an optimal solution in step 21 or step 29.

(i) If step 25 is executed infinitely many times and the sequence $(d_i, \lambda_i)$ converges to some limit point $(d_*, \lambda_*)$, then $d_*$ is the unique optimal solution of $QP$ (1.1).

(ii) If step 25 is executed finitely many times, the algorithm eventually stays in the iterative linear solver loop (steps 10–23) for some active set $A$. If the corresponding sequence of iterative linear solver iterates $(d^F_j, \lambda_j)$ converges to some limit point $(d^F_*, \lambda_*)$, then $d_*$ defined by $d^F_*=d^F_{\ast}$ and $d^\ell_*=\ell^A$ is the unique optimal solution of $QP$ (1.1) with corresponding optimal multipliers $\lambda_*=\lambda_{\ast}$.

(iii) Suppose that $(d_*, \lambda_*, \mu_*)$ is the unique optimal primal-dual solution of $QP$ (1.1) with active set $A_*=A(d_*)$, that strict complementarity holds, and that $A^T$ and $A^T$ with $F_* = A_*^C$ have full row rank. Then, if $(d_1,\lambda_1)$ is sufficiently close to $(d_*, \lambda_*)$, Algorithm 2 eventually remains in the iterative linear solver loop with active set $A = A_*$, and the iterative linear solver iterates $(d^F_j, \lambda_j)$ converge to $(d^F_*, \lambda_*)$.

**Proof.** (i) If Algorithm 2 executes the iterative refinement update in steps 4 and 25 infinitely many times, we can argue as in the proof of Theorem 4.3(i) that the limit point $(d_*, \lambda_*)$ satisfies the optimality conditions for $QP$ (1.1).

(ii) If step 25 is executed only a finite number of times, the algorithm eventually performs only iterative linear solver iterations in the $j$-loop with some fixed active set $A$. Because step 18 is not reached, the test in step 17 is not true, and therefore $d^F_j \geq \ell^F$ for all $j$, so that in the limit $d^F_j \geq \ell^F$ and hence by construction $d_* \geq \ell$. Similarly, because step 14 is not reached, we have $z^A_j = 0$ for all $j$.

Furthermore, the preconditioning $QP$ (5.4) is solved in each iteration, and its optimality conditions imply
\begin{equation}
\begin{bmatrix}
W_{FF} & W_{FA} (A^T)^T \\
W_{AF} & W_{AA} (A^T)^T - I
\end{bmatrix}
\begin{bmatrix}
z^F_j \\
(z^A_j)'
\end{bmatrix}
= \begin{bmatrix}
-g^A + W_{AF}d^F_j + W_{AA}\ell^A + (A^T)\lambda_j \\
(s_j)
\end{bmatrix},
\end{equation}
where $\mu^A_j \geq 0$ are the multipliers for the bound constraints (5.4c). Because we assume that the iterates converge, it follows from Assumption 1(i) that $(z^F_j, z^A_j)$ and residuals
(r_j, s_j) converge to zero, and the optimality conditions above become in the limit (5.7)

\[
\begin{bmatrix}
\bar{W}^{FF} & \bar{W}^{FA} (\bar{A}^T 0 \\
\bar{W}^{AF} & \bar{A} A^T -I \\
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0 \\
\mu^A \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

with \( \mu^A \geq 0 \) since \( z^A_j = 0 \) for all \( j \).

Finally, because we assume that the iterates converge, it follows from Assumption 1(i) that \((\bar{d}^F_j, \bar{\lambda})\) satisfy (5.1). Together with (5.7), this implies that the optimality conditions (4.10) for QP (1.1) are satisfied by \((\bar{d}_*, \bar{\lambda}_*)\).

(iii) Note that \((\bar{d}^F_j, \bar{\lambda}_j) = (\bar{d}^F_j, \bar{\lambda}_*)\) is the solution of (5.1). Due to strict complementarity, we have \(d^F_j > \ell^F_j\), and from Assumption 1(iii) we then have that \(d^F_j > \ell^F_j\) for all \( j \) if \((\bar{d}^F_j, \bar{\lambda}_j)\) is sufficiently close to \((\bar{d}^F_j, \bar{\lambda}_*)\). Similarly, \(\mu^A_* = g^A_* + W^A^T d^F_* + W^A^T \ell^A + (A^T \bar{\lambda}_*)^T \bar{\lambda}_y > 0\).

Because \(\bar{W}\) is positive definite and \(\bar{A}^T\) has full row rank, the matrix in (5.6) is nonsingular. Therefore, we then havethe result from (5.6) and (5.5) that \(\mu^A_j > 0\) for all \( j \) if \((\bar{d}^F_j, \bar{\lambda}_j)\) is sufficiently close to \((\bar{d}^F_j, \bar{\lambda}_*)\), and due to complementarity, \(z^A_j = 0\) for all \( j \).

In summary, we conclude that Algorithm 2 remains in the iterative linear solver loop if it is invoked at an iterate \((d_i, \lambda_i)\) \(\in B_{\epsilon_1}\) for a sufficiently small \(\epsilon_1 > 0\) in step 9, because then the conditions in steps 13 and 17 will never be met.

Let \(\epsilon > 0\) and \(c_1 > 0\) be the constants from Lemma 4.2. Set \(\epsilon_2 = \min\{\epsilon, \epsilon_1\}/\max\{1, c_1\}\) and let \((d_1, \lambda_1) \in B_{\epsilon_2} \subseteq B_{\epsilon}\). Then \(A(d_2) = A_*\) and \((d_1, \lambda_1) \in B_{\epsilon_2} \subseteq B_{\epsilon}\). Applying this argument a second time, we conclude that Algorithm 2 remains in the iterative linear solver loop if \(\epsilon_1 > 0\) in step 9, because then the conditions in steps 13 and 17 will never be met.

Remark 1. For the discussion above, we assumed that the matrix in (5.2) is nonsingular. However, it is possible that the solution of (4.7) is degenerate and that the gradients of the active constraints are linearly dependent. In this case, the active set \(A_*\) identified in step 5 leads to a preconditioning system (5.2) with a singular matrix because \(\bar{A}^T\) does not have full row rank. Nevertheless, as mentioned at the end of section 2, active-set QP solvers usually maintain a “working set” \(W\) of linearly independent constraints that identify the optimal solution. Using this working set in place of the active set \(A_*\) in step 5, we are guaranteed to always obtain a nonsingular preconditioning system (5.2) if \(F = W^F\). In addition, if we assume that the QP solver returns the same working set whenever the active set has not changed during a hot-start for (4.7), Algorithm 2 will still detect when the active set remains unchanged and enter the iterative linear solver loop in step 6.

Remark 2. The convergence theorems of sections 4 and 5 were established under the assumption that the Hessian matrices \(W\) and \(\bar{W}\) are positive definite. It can be verified that the results still hold if this assumption is replaced by the requirement that the projection of \(W_{FF}^F\) onto the null space of \(A^F\) is positive definite, and that the projection of \(\bar{W}_{FF}\) onto the null space of \(\bar{A}^F\) is positive definite for any free set \(F\) corresponding to the optimal solutions of the QPs (4.7) and (5.4) encountered during the algorithm. In that case, the optimal solutions of the QPs are still unique, the
matrices in (4.13) and (5.6) are still nonsingular, and the sensitivity results required in the proof of Lemma 4.2 still apply.

6. Numerical results. To examine the practical performance of the proposed approach, a prototype implementation of Algorithm 2, which we will refer to as iQP (inexact QP solver), was created in MATLAB R2012b. SQMR [17] was chosen as the iterative linear solver for the inner loop, because it exploits the symmetry of the matrix and allows indefinite preconditioners. For completeness, the detailed description of Algorithm 2 using SQMR is provided in Appendix A. We note that SQMR does not have theoretical convergence guarantees; in our implementation, SQMR is simply restarted if it breaks down, but this fall-back was triggered in our experiments very rarely. The QPs (4.7) and (5.4) were solved using the open-source active-set parametric QP solver qpOASES [11, 12]. All experiments were performed on an 8-core Intel-i7 3.4GHz 64bit Linux server with 32GB RAM. MATLAB was set to use only a single computational thread.

We present two sets of numerical experiments. In section 6.1, Algorithm 2 is used to solve a sequence of QPs that arise in certain nonlinear model predictive control (NMPC) applications. In section 6.2, a sequence of randomly perturbed quadratically-constrained quadratic programs (QCQPs) is solved.

The goal of these experiments is twofold. First, we explore the reliability of the new QP method in practice, given that convergence is not guaranteed. Second, we compare the performance of the iQP method (the hot-start approach) with that of a standard active-set solver, qpOASES. We refer to the latter as the warm-start approach to indicate that the solution of a new QP can be started from the optimal active set of the previous QP but that the KKT matrix in (2.2) has to be factorized from scratch because the entries changed. Because qpOASES uses dense linear algebra in its current implementation, our experiments are carried out for problems with dense matrices.

Whether the new method requires overall less computation time for the solution of a new QP in a sequence depends on a number of factors. The warm-start approach requires the factorization of the KKT matrix in (2.2); this costs roughly $O((n_F)^3)$ floating-point operations for dense matrices, where $n_F$ is the number of free variables. In addition, for each iteration of the active-set QP solver, in which one variable leaves or enters the active set, the linear system (2.2) is solved twice, and the factorization of the KKT matrix is updated for a new active set; this requires roughly $O((n_F)^2)$ operations. On the other hand, the hot-start approach does not require a factorization with work $O((n_F)^3)$, but the number of solves of the linear system (2.2) increases because one or two hot-started QPs have to be solved per iQP iteration, each of which might require several active-set changes, particularly in the first iQP iterations. Therefore, whether the new approach requires less computational effort depends on the number of iQP iterations and the relative cost of factorizing the KKT matrix in (2.2) versus the backsolve given the factorization.

In some applications, the computation of the matrices $A$ and $W$ in (1.1) dominates the computational time. This is, for example, the case when the constraints arise from the integration of differential equations, as in the NMPC context (see section 6.1.3). Then, single matrix elements cannot be accessed individually, and the effort for computing the entire $A$ matrix is equivalent to evaluating $A$ rowwise (or columnwise) by computing $n$ products of $A$ (or $m$ products of $A^T$) with unit vectors. Each product involves the computation of a directional derivative of the solution of the differential equation. Therefore, we also report the number of matrix-vector products involving both $A \cdot v$ and $A^T \cdot v$ in our statistics. If this count is significantly smaller than $n$ or
We denote by \( (6.1h) \) \( m \) over the entire execution of an \( iQP \) run, a large reduction in overall computation time can be expected compared to the warm-start approach which requires the full matrix \( A \).

6.1. QPs from nonlinear model-predictive control. In this section, we investigate the performance of \( iQP \) on a sequence of QPs arising in an NMPC application.

6.1.1. Chain of point masses problem. Our NMPC case study involves a motion control problem for a chain of \( N_{PM} \) free point masses, indexed by \( 1 \leq i \leq N_{PM} \), that are connected by springs and subject to gravity. An additional point mass, indexed by 0, is fixed at the origin. Point mass positions at time \( t \) are denoted by \( x_i(t) = (x_{i,x}(t), x_{i,y}(t), x_{i,z}(t)) \in \mathbb{R}^3 \) and velocities by \( v_i(t) \in \mathbb{R}^3 \). Starting with initial conditions \( x_i(0) = (7.5i/N_{PM}, 0, 0)^T \), \( v_i(0) = (0, 0, 0)^T \), the point masses are accelerated by gravity and the chain’s springs expand. The dynamic model is free of friction such that, once accelerated by gravity, it does not return to rest without appropriate application of external forces. The velocity \( v_{N_{PM}}(t) \) of the final point mass may be controlled through \( u(t) = (u_x(t), u_y(t), u_z(t)) \in \mathbb{R}^3 \). The goal of the controller is to determine velocities \( u(t) \) for the final point mass that bring the chain to rest. This optimal control problem (OCP) can be written as

\[
\begin{align*}
\min_{x(t), v(t), u(t)} & \quad \int_0^T w_{vel} \sum_{i=1}^{N_{PM}} \|v_i(t)\|^2_2 + w_{pos} \|x_{N_{PM}}(t) - x_e\|^2_2 \\
& \quad + w_{u} \|u(t)\|^2_2 + w_{al} \|u(t)\|^2_2 \, dt \\
\text{s.t.} & \quad \dot{x}_i(t) = v_i(t), \quad t \in [0, T], \ 1 \leq i < N_{PM}, \\
& \quad \dot{v}_i(t) = (F_{i+1}(t) - F_i(t)) \cdot N_{PM}/m - g, \quad t \in [0, T], \ 1 \leq i < N_{PM}, \\
& \quad \dot{x}_{N_{PM}}(t) = u(t), \quad t \in [0, T], \\
& \quad x(0) = \hat{x}_0, \\
& \quad u(t) \in [-1, 1]^3, \quad t \in [0, T], \\
& \quad x_{i,x}^{\min} \leq x_{i,x}(t) \leq x_{i,x}^{\max}, \quad t \in [0, T], \ 1 \leq i < N_{PM}, \\
& \quad x_{i,z}^{\min} \leq x_{i,z}(t) + u_{al,i}(t), \quad t \in [0, T], \ 1 \leq i < N_{PM}, \\
& \quad u_{al}(t) \geq 0, \quad t \in [0, T].
\end{align*}
\]

We denote by \( F_i(t) \) the forces \( F_i(t) := (x_i(t) - x_{i-1}(t)) \cdot k (N_{PM} - l_i/\|x_i(t) - x_{i-1}(t)\|_2) \). The slack variables \( u_{al,i}(t) \) penalize a violation of the lower bound on \( x_{i,z}(t) \). Characteristics and weights are given in Table 1. This problem has been considered in similar form in, e.g., [21].

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Symbol & Value & Unit \\
\hline
\( g \) & \((0, 0.9, 8.1)^T \) & \( \text{m/s}^2 \) \\
\( k \) & 0.1 & \( \text{N/m} \) \\
\( l_i \) & 0.55 & \( \text{m} \) \\
\( m \) & 0.45 & \( \text{kg} \) \\
\( x_e \) & \((7.5, 0, 0)^T \) & \( \text{m} \) \\
\hline
\end{tabular}
\end{table}
6.1.2. Nonlinear model-predictive control. To simplify the notation, we rewrite (6.1b)–(6.1e) as
\[
\dot{y}(t) = D(y(t), u(t)), \quad t \in [0, T], \quad y(0) = \hat{y}_0.
\]
In the online NMPC setting, one considers a sequence of sample times $\tau^k$, indexed by $k$. The state $\hat{y}_0(\tau^k)$ of a physical system is monitored (sampled) at $\tau^k$. Following the idea of real-time iterations \cite{10}, the optimal control action is computed as the solution of a feedback QP; see section 6.1.3. From this solution, the beginning of the optimal control action $u^*(\tau^k)$ is applied to the system. After the feedback interval $\Delta \tau$ has elapsed, the system state $\hat{y}_0(\tau^{k+1})$ is remeasured at $\tau^{k+1} = \tau^k + \Delta \tau$ and the feedback QP is resolved with the new value of $\hat{y}_0$. Hence, the number of NMPC samples is equal to $N_{\text{QP}}$, the number of QPs solved.

In order to simulate the change of the point masses process from one sample time $\tau^k$ to the next $\tau^{k+1}$ in our experiments, the forward problem (6.1b)–(6.1d) is solved, starting in the previous initial value $\hat{y}_0(\tau^k)$ and applying the most recent feedback control $u(t)$ for the duration of the sampling interval, $t \in [t_0, t_1]$. The state at the end of this simulation is then taken as the (unperturbed) initial conditions $\hat{y}_0(\tau^{k+1})$ for the next sample.

6.1.3. Feedback QP problem. In our experiment, we follow the direct multiple shooting approach \cite{7, 22} to obtain the feedback QP from the OCP (6.1) as follows. We choose an equidistant time discretization $0 = t_0 < t_1 < \cdots < t_{N_{\text{MS}} - 1} < t_{N_{\text{MS}}} = T$ of the prediction horizon $[0, T]$ into $N_{\text{MS}}$ shooting intervals $[t_j, t_{j+1}]$, $0 \leq j < N_{\text{MS}}$. We introduce control parameters $q_j \in \mathbb{R}^n$ with $n_a = n_u + 3 + (N_{\text{PM}} - 1)$ for a piecewise constant control discretization,
\[
\begin{align*}
(u(t), u_{\text{sl}}(t))|_{t \in [t_j, t_{j+1}]} = q_j & \quad \text{for } 0 \leq j < N_{\text{MS}}.
\end{align*}
\]
In addition, we apply the multiple shooting state parametrization
\[
\dot{y}(t) = D(y(t), q_j), \quad t \in [t_j, t_{j+1}], \quad y(t_j) = s_j,
\]
that decouples the forward problem (6.2) into $N_{\text{MS}}$ initial value problems. In order to ensure consistency of the optimal solution, we introduce the additional matching conditions
\[
\begin{align*}
y(t_{j+1}; t_j, s_j, q_j) - s_{j+1} = 0, & \quad 0 \leq j < N_{\text{MS}}.
\end{align*}
\]
Herein, $y(t_{j+1}; t_j, s_j, q_j)$ denotes the solution of (6.3) on $[t_j, t_{j+1}]$ evaluated at $t_{j+1}$ when started with initial value $s_j$, applying the control $q_j$. Inequality path constraints (6.1g) and control bounds (6.1h) are enforced on the shooting grid $\{t_j\}_{0 \leq j \leq N_{\text{MS}}}$, resulting in constraints of the form
\[
0 \leq r_j(s_j, q_j), \quad 0 \leq j \leq N_{\text{MS}}.
\]
Here, we set $q_{N_{\text{MS}}} = q_{N_{\text{MS}} - 1}$ for simplicity of notation in (6.5). In summary, this discretization and parametrization transforms problem (6.1) into a discrete-time control problem that is a finite-dimensional NLP.
The local quadratic approximation of this NLP about a reference point \((\bar{s}, \bar{q})\) is

\[
\begin{align*}
(6.6a) & \quad \min_{z_j=(s_j, q_j)} \sum_{j=0}^{N_{\text{MS}}} \frac{1}{2} z_j^T W_j z_j + q_j^T z_j \\
(6.6b) & \quad s_{j+1} = D_j s_j + E_j q_j + f_j, \quad 0 \leq j < N_{\text{MS}}, \\
(6.6c) & \quad s_0 = \hat{y}_0, \\
(6.6d) & \quad q_j \in [-1, 1]^3, \quad 0 \leq j \leq N_{\text{MS}}, \\
(6.6e) & \quad 0 \leq P_j s_j + Q_j q_j + p_j, \quad 0 \leq j \leq N_{\text{MS}},
\end{align*}
\]

with the Hessian blocks \(W_j\), the gradient of the objective parts \(g_j\), constraint matrices \(D_j = \nabla_s y(t_{j+1}; t_j, s_j, q_j)\), \(E_j = \nabla_q y(t_{j+1}; t_j, s_j, q_j)\), \(P_j = \nabla_s r_j(s_j, q_j)\), and \(Q_j = \nabla_q r_j(s_j, q_j)\), and constraint vectors \(f_j = y(t_{j+1}; t_j, s_j, q_j)\) and \(p_j = r_j(s_j, q_j)\). The blocks \(W_j\) are chosen as Gauss–Newton approximations to exploit the least-squares nature of the tracking NMPC objective function (6.1a). The “expensive” constraint Jacobians \(D_j, E_j\) and the Hessians \(W_j\) are usually dense and require the numerical computation of sensitivities of the solution of the initial value problem (6.3) with respect to all independent variables \(s_j\) and \(q_j\), and capture the ODE dynamics of the process. Typically, computing the sensitivity of \(y(t_{j+1}; t_j, s_j, q_j)\) with respect to a single variable \(s_j^{(i)}\) or \(q_j^{(i)}\) is about as time-consuming as a forward integration [2]. As a consequence, the computation of \(D_j, E_j\), and \(W_j\) can become the computational bottleneck and can take up more than 90 percent of the CPU time [21]. To address this, [6] proposed an NMPC algorithm named “Mode C” that solves a single QP (4.11), i.e., it performs one iteration of Algorithm 1. The iQP approach proposed in this article improves over this idea by employing a preconditioner and performing multiple iterations to compute an improved solution.

6.1.4. Results. For the NMPC computations, the software package from [20] is used. The ODE system (6.3) is integrated with a fourth-order Runge–Kutta method with 20 equidistant time steps per multiple shooting interval. Analytic derivatives of the model equations are available such that sensitivities of the discretized ODE system, computed according to the principle of internal numerical differentiation [5], are available with machine precision.

We consider the NMPC scenario for chains with \(N_{\text{PM}} \in \{6, 8, 10, 12, 14\}\) point masses and for a prediction horizon of \(T = 8\) seconds discretized into \(N_{\text{MS}} \in \{15, 20\}\) intervals. We give feedback every \(\Delta \tau = T/N_{\text{MS}} \in \{0.5333, 0.4\}\) seconds and run this scenario for \(\tau_{\text{max}} = 30\) seconds, computing \(N_{\text{QP}} = [\tau_{\text{max}} N_{\text{MS}} / T] \in \{57, 76\}\) samples. This duration and sampling rate was sufficient for the NMPC controller to successfully settle the system in all investigated scenarios. The resulting dimensions are listed in Table 2, where \(n_y\) is the number of state variables, \(n_u\) is the number of control and slack variables, and \(n\) and \(m\) are the number of QP variables and linear constraints (excluding bound constraints). We obtain the initial QP (1.2) as the quadratic approximation (6.6) at the steady state of the system, which is obtained by setting \(u(t) \equiv 0\) and \(y(t) \equiv \hat{y}_0\) such that the chain is at rest, satisfying \(D(\hat{y}_0, 0) \equiv 0\).

In Table 2, we report the performance of the iQP algorithm on the ten QP sequences. The stopping criterion for Algorithm 2 was defined as \(\Phi(d, \lambda) \leq 10^{-8}\) with \(\Phi(d, \lambda)\) defined in (4.9). All QPs were solved successfully by iQP, with the exception of one QP in the \(N_{\text{MS}} = 20, N_{\text{PM}} = 6\) series for which the maximum number of 100 iterations was exceeded. The performance metric for this experiment is the number of matrix-vector products with the constraint matrix \(A\) in (1.1) and its transpose, which
Table 2

iQP statistics for the NMPC case study. Columns one and two show the choice of \(N_{MS}\) (number of multiple shooting intervals) and \(N_{PM}\) (number of point masses) for an instance of the OCP (6.1). Columns three and four show the number of differential states and the number of controls present in the instance. Columns five and six show the number \(n\) of variables and the number \(m\) of linear constraints (excluding bound constraints) present in the QPs (6.6). Columns seven and eight show the number of QPs obtained and the number of QPs solved successfully by iQP. The final four columns show the number of matrix-vector products required by iQP, compared to the number that would be required to once compute a full KKT matrix.

<table>
<thead>
<tr>
<th>OCP dimensions</th>
<th>QP dims</th>
<th># of QPs</th>
<th>Matrix-vector products</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N_{MS})</td>
<td>(n_y)</td>
<td>(n_u)</td>
<td>(n)</td>
</tr>
<tr>
<td>15</td>
<td>6</td>
<td>33</td>
<td>8</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
<td>45</td>
<td>10</td>
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<tr>
<td>15</td>
<td>10</td>
<td>57</td>
<td>12</td>
</tr>
<tr>
<td>15</td>
<td>12</td>
<td>69</td>
<td>14</td>
</tr>
<tr>
<td>15</td>
<td>14</td>
<td>81</td>
<td>16</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>33</td>
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<tr>
<td>20</td>
<td>14</td>
<td>81</td>
<td>16</td>
</tr>
</tbody>
</table>

Fig. 2. Number of matrix-vector products and active set changes over the course of 57 solved QPs for the \(N_{PM} = 6, N_{MS} = 15\) chain simulation.

are computationally expensive because these involve the sensitivity matrices \(D_j\) and \(E_j\) and require computations by the ODE solver. The minimum, maximum, and mean of the total number of products required by iQP during the solution of the QPs (4.7) and (5.4) is compared with the equivalent number of matrix-vector products that are necessary to compute the entries of the almost block-diagonal matrix \(A_k\). Here, we assume that each of the \(N_{MS}\) blocks in \(A\) can be obtained by \(n_y\) products of the transpose of the sensitivity matrices with unit vectors. As can be seen, iQP reduces this effort by a factor of up to 2.4 on average.

As the physical system settles and gets closer to the desired steady-state solution, the differences between the QPs from one sample time to the next become smaller, and iQP requires fewer iterations. This can be seen in Figure 2, which illustrates the diminishing number of active set changes and matrix-vector products for an iQP run over the QP sequence in a typical instance.

6.2. Solving sequences of similar NLPs with SQP. As briefly discussed in the introduction, a sequence of QPs with similar data also arises when the SQP
algorithm is applied to a sequence of similar NLPs. In this section, we consider the sequential solution of quadratically constrained quadratic problems, which we will refer to as QCQP\((t)\) with \(t = 0, 1, 2, \ldots,\) of the form

\[
\begin{align}
(6.7a) & \quad \min_{x \in \mathbb{R}^n} \frac{1}{2} x^T H_0 x + (q_0^t)^T x + r_0^t \\
(6.7b) & \quad \text{s.t. } \frac{1}{2} x^T H_j x + (q_j^t)^T x + r_j^t \leq 0, \quad 1 \leq j \leq m, \\
(6.7c) & \quad x \geq 0.
\end{align}
\]

6.2.1. Experimental setup. For a chosen problem size of \(n\) variables and \(m\) constraints, the initial problem, indexed by \(t = 0\), is generated using the following steps [27]:

1. Choose optimal solution values \(x_* \sim \mathcal{U}(-1,1)^n\), \(\lambda_* \sim \mathcal{U}(0,1)^m\), and \(\mu_* \sim \mathcal{U}(0,1)^n\), where \(\mathcal{U}(a,b)\) is the uniform distribution on the interval \([a,b]\).

2. Adapt the solution so that the first \(\kappa_x\) variable bounds and the first \(\kappa_c\) inequality constraints are active: Reset the first \(\kappa_x\) elements in \(x_*\), the last \(n - \kappa_x\) elements in \(\mu_*\), and the last \(m - \kappa_c\) elements in \(\lambda_*\) to zero.

Here, \(0 \leq \kappa_c \leq m\) and \(0 \leq \kappa_c \leq n\) denote the number of inequality constraints (6.7b) and variable bound constraints (6.7c) that are active at the optimal solution. For our experiments, we choose \(\kappa_c = \lfloor 1/3m \rfloor\) and \(\kappa_x = \lfloor 1/3n \rfloor\).

3. For \(j = 0, \ldots, m\), generate random symmetric positive definite matrices \(H_j\) \((j = 0, \ldots, m)\) of dimension \(n \times n\) with condition number 1000. For our experiments, we use the MATLAB function \texttt{sprandsym}(n,0.2,1e-3,1).

4. For \(j = 1, \ldots, m\), choose \(q_0^j \sim \mathcal{U}(-1,1)^n\) and set \(r_0^j := -\frac{1}{2} x_*^T H_j x_* + (q_0^j)^T x_*\).

Add a uniform random number from \(\mathcal{U}(0,1)\) to each of the last \(m - \kappa_c\) elements in \(r_0^j\) to make the corresponding constraints inactive.

5. Set \(q_0^j := -((H_0 + \lambda_*^{(j)} H_j)x_* + \sum_{j=1}^m \lambda_*^{(j)} q_0^j - \mu_*)\) and \(r_0^j := -\frac{1}{2} x_*^T H_0 x_* + (q_0^j)^T x_*\).

It can easily be verified that then \(x_*\) is the optimal solution of (6.7) for \(t = 0\) and that \(\lambda_*\) and \(\mu_*\) are the corresponding multipliers. By construction, strict complementarity holds, and the objective and constraint functions are convex. Note that this problem can be reformulated into the standard form (1.4) by introducing slack variables.

From the QCQP\((0)\) data, we generate the nearby problem instance QCQP\((t)\) by perturbing each entry in the problem data \(q_j^t\) and \(r_j^t\) via

\[
(q_j^t)^{(i)} \sim \mathcal{N} \left( (q_j^0)^{(i)}, \sigma^2 \right), \quad (r_j^0)^{(i)} \sim \mathcal{N} \left( (r_j^0)^{(i)}, \sigma^2 \right), \quad i = 1, \ldots, n \text{ and } j = 0, 1, \ldots, m.
\]

Here, \(\mathcal{N}(\mu, \sigma^2)\) denotes the normal distribution with mean \(\mu\) and variance \(\sigma^2\), and \(\sigma > 0\) is a fixed parameter controlling the size of the perturbation.

These QCQPs are solved using the MATLAB implementation “p-sqp,” developed by Frank E. Curtis, of the \(\ell_1\)QP method proposed in [8] with minor modifications. In this algorithm, at an SQP iterate \((x_k, \lambda_k)\), the search direction for the line search is obtained as the optimal solution of the \(\ell_1\)QP

\[
\begin{align}
(6.9a) & \quad \min_{\Delta x, p, s} \rho_k \nabla f(x_k)^T \Delta x + \frac{1}{2} \Delta x^T W_k \Delta x + c^T p \\
(6.9b) & \quad \text{s.t. } \nabla c(x_k)^T \Delta x + c(x_k) + s - p = 0, \\
(6.9c) & \quad x_k + \Delta x \geq 0, \\
(6.9d) & \quad s, p \geq 0.
\end{align}
\]
where \( e = (1, \ldots, 1)^T \), \( \rho_k \geq 0 \) is the current value of the (inverse of the) penalty parameter, and \( W_k = \rho_k \nabla^2 f(x_k) + \sum_{j=1}^m \lambda_k^{(j)} \nabla^2 c^{(j)}(x_k) \). Due to the convexity assumption, \( W_k \) is always positive definite, because the vector \((\rho_k, \lambda_k)\) remains nonnegative and nonzero throughout the optimization. The details of the SQP method are not relevant here; the interested reader is referred to [8]. We only point out that the \( \ell_1 \) QP \( (6.9) \) is always feasible by construction. In addition, at least one of \( s^{(j)} \) and \( p^{(j)} \) is zero for all \( j = 1, \ldots, m \) at the optimal solution of \( (6.9) \). Using this observation, it is not difficult to show that then the projection of the Hessian matrix of \( (6.9) \) onto the null space of the gradients of the active constraints is positive definite, so that the conditions described in Remark 2 hold.

In our experimental setup, we initially solve QCQP(0) with the SQP method using a standard active-set QP solver \( \text{qpOASES} \) in our context) and then “fix” the QP \( (6.9) \) corresponding to the instance QCQP(0) at the returned solution \( x^* \) and \( \lambda^* \) as the initial QP \( (1.2) \), i.e., \( A_0 \) and \( W_0 \) are chosen to be the matrices corresponding to \( x^* \) and \( \lambda^* \). The internal state of the QP solver is also stored. We then apply the \( \ell_1 \) QP algorithm to the perturbed QCQP(t) problems with \( t = 1, 2, 3, \ldots \), using \( x^* \) and \( \lambda^* \) as initial iterates. The termination tolerance for the \( \ell_1 \) QP algorithm is set to \( 10^{-6} \).

The QPs \( (6.9) \) are solved using our \text{iqP} implementation of Algorithm 2. At the beginning of each SQP run, the QP solver for \( (1.3) \) is restored to the internal state corresponding to \( (x^*, \lambda^*) \), and subsequently only hot-starts are used for any solution of \( (4.7) \) and \( (5.4) \) required for Algorithm 2. In each SQP iteration \( k \), Algorithm 2 is terminated when the termination criterion

\[
\Phi((\Delta x, s, p), \lambda) \leq \min\{10^{-8}, 10^{-5} \epsilon_k\}
\]

is satisfied, where \( \epsilon_k \) is the KKT error for QCQP(t) in iteration \( k \), and \( \Phi \) is defined in \( (4.9) \).

This tight tolerance is necessary because the convergence analysis for \( \ell_1 \) QP method in [8] assumes the exact solution of \( (6.9) \), and \text{p-sqp} frequently fails to converge if less accurate solutions are returned. The experiments below show that even such highly accurate solutions are obtained by Algorithm 2 with a reasonable amount of work. If Algorithm 2 fails to satisfy condition \( (6.10) \) within \( 100 \) \text{iqP} iterations in some SQP iteration \( k \), the SQP algorithm is terminated with an error message.

**6.2.2. Results.** The detailed results of our numerical experiments are reported in Appendix B. A total of 64 combinations of sizes and perturbation levels \( \sigma \) were considered: The numbers of variables were chosen as \( n \in \{50, 200, 500, 1000\} \), and the numbers \( m \) of inequality constraints took the values \( 20\%n, 50\%n, 80\%n \), and \( 150\%n \). The perturbations were chosen as \( \sigma \in \{0.01, 0.05, 0.1, 0.2\} \). For each such combination, one QCQP(0) was generated, and then 10 perturbed instances were solved (only 3 perturbed instances for \( n = 1000 \) due to the excessive computation times). The values reported in Appendix B are the averages over those 10 (or 3) runs.

The SQP algorithm was run twice on each instance, once with the standard active-set QP solver \text{qpOASES} and once with the new method \text{iqP} to solve the step computation QP \( (6.9) \). Except for eight out of about 500 successfully solved instances, the number of SQP iterations was identical for both QP solvers.

Table 3 presents a summary of the performance of \text{iqP}. It is noteworthy that \text{iqP}, despite the lack of a convergence guarantee, is able to solve most of the QPs and exhibits a considerable level of reliability except when the problem perturbation becomes large. Furthermore, the number of \text{iqP} iterations is on average only between
Statistical observations for the QCQP experiments.

<table>
<thead>
<tr>
<th>Size of perturbation $\sigma$</th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
<th>0.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Successfully solved QCQPs</td>
<td>99.3%</td>
<td>99.3%</td>
<td>97.7%</td>
<td>42.9%</td>
</tr>
<tr>
<td>Average number of QP iterations</td>
<td>4.9</td>
<td>8.5</td>
<td>12.5</td>
<td>20.6</td>
</tr>
<tr>
<td>Average change in active set for (6.7b)</td>
<td>5.3%</td>
<td>19.0%</td>
<td>25.7%</td>
<td>32.4%</td>
</tr>
<tr>
<td>Average change in active set for (6.7c)</td>
<td>3.4%</td>
<td>9.5%</td>
<td>13.8%</td>
<td>19.2%</td>
</tr>
</tbody>
</table>

Fig. 3. Relative performance of qpOASES vs. iQP. For a given number of constraints $m$, number of variables $n$ and perturbation $\sigma$, the graphs show the ratio $r = \text{metric}_{qpOASES}/\text{metric}_{iQP}$, where “metric” is the average CPU time or number of matrix-vector products.

5 and 20 per QP, even though the level of accuracy is rather tight. (The right-hand side in (6.10) is between $10^{-8}$ and $10^{-11}$.) We also observe that the sets of inequality constraints (6.7b) and variable bounds (6.7c) that are active at the optimal solution of the initial QCQP(0) and the perturbed QCQP($t$) are significantly different, showing that the problem perturbations are non-trivial.

Figures 3(a) and 3(b) compare the performance of qpOASES and iQP in terms of CPU time. As the problem size increases, the new method becomes increasingly faster compared to the standard approach; for $n = 1000$, we see a reduction of up to two orders of magnitude in CPU time. However, we point out that in this experiment all matrices in (1.1) are dense, and the balance is likely to shift in favor of qpOASES if sparse linear algebra methods can be used.

We also compare the number of matrix-vector products involving $\nabla c(x_k)$ and $\nabla c(x_k)^T$ required by the algorithm. This is relevant if the evaluation of the constraint Jacobians is the bottleneck of the computation as in section 6.1. Figures 3(c) and 3(d) show a significant improvement, with a reduction of up to more than one order.
Table 4

Number of \texttt{qpOASES} pivots in an iQP iteration within the given SQP iteration. The superscript \(+ (\pm)\) indicates that the inner SQMR loop in Algorithm 2 was started (started and immediately terminated) in an iteration.

<table>
<thead>
<tr>
<th>iQP iteration</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>SQP iteration</td>
<td>0</td>
<td>13</td>
<td>6</td>
<td>5(\pm)</td>
<td>5</td>
<td>2(\pm)</td>
<td>1</td>
<td>2</td>
<td>0(+)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>2</td>
<td>0(+)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0(+)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3-7</td>
<td>1</td>
<td>0(+)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

of magnitude for the large instances. Here, the number of matrix-vector products is obtained by counting the products with both \(\nabla c(x_k)\) and \(\nabla c(x_k)^T\) during an iQP run. For the \texttt{qpOASES} case, we consider \(m\) matrix-vector products with \(\nabla c(x_k)\) to be equivalent to the computation of the full matrix \(\nabla c(x_k)\), since all matrix elements can be obtained by products of \(\nabla c(x_k)\) with the \(m\) unit vectors.

Finally, to see the progression over an entire SQP run, we report one typical case in more detail. Table 4 lists the total number of pivots taken by \texttt{qpOASES} when solving the QPs (4.7) or QP (5.4) for each iQP iteration over the course of the SQP algorithm. Most pivots are taken in the first iQP iteration of the first SQP iteration, indicating that the active set changes significantly compared to the initial QP. In the later iQP iterations, in which SQMR updates are performed, no pivots are required because then the active set remains constant.

7. Conclusions. We proposed a new QP algorithm that uses hot-starts of an active-set QP solver from a previously solved initial QP in order to accelerate the solution of a similar QP. The numerical study showed that this approach can reduce the computational effort when a sequence of similar QPs or NLPs is solved. Our approach has two advantages.

First, when the computation of the constraint matrix of the QP requires expensive calculations, such as the integration of differential equations, the evaluation of the full constraint matrix can be avoided. In that case, only matrix-vector products (obtained using adjoint calculations or automatic differentiation techniques) are required. This benefit was demonstrated on a nonlinear model-predictive control example.

Second, speedup can be obtained when, for each new SQP iteration during the solution of an NLP, the factorization of the KKT matrix inside an active-set QP solver is replaced by a sequence of hot-starts. This observation was made for a set of randomly perturbed NLPs with dense derivative matrices. It remains a subject of future research whether this advantage is also observed when sparse linear algebra techniques can be used. Furthermore, we postulate that additional computation time could be saved if the SQP algorithm is designed to handle inexact QP solutions so that our method could be terminated after fewer iterations. Such a candidate algorithm has been proposed in [9].

The proposed algorithm is proven to converge if it is started sufficiently close to a nondegenerate QP solution. However, in general, the method may diverge or cycle, like any iterative refinement procedure for linear systems. One premise of the present work is that a black-box QP solver can be used in this framework and that this QP solver is responsible for handling the update of the active set. It appears difficult or impossible to design a globally convergent variant of the proposed algorithm without explicitly managing the active set. Nevertheless, the numerical results show that the
method is robust to moderate changes of the QP data. In a practical setting, one could attempt to solve a QP with the proposed method, and if cycling or divergence is observed, the QP could be solved with a regular active-set QP solver. This new QP solution may then be used as the new initial QP.

Appendix A. Detailed description of Algorithm (2) for SQMR.

**Algorithm 3. Solving QP (1.1) using hot-starts for QPs (4.7) and (5.4), accelerated by SQMR.**

1: Given: Initial iterates $d_i \geq \ell$ and $\lambda_i$.
2: Initialize: $A_0 = \{l | d_i^{(0)} = \ell(l)\}$ and $\text{ref\_flag} \leftarrow \text{false}$.
3: for $i = 1, 2, 3, \ldots$ do
4:     Solve QP (4.7) to obtain optimal solution $p_i$ with optimal multipliers $p_i^\lambda$.
5:     Determine the active set $A_i = \{l | d_i^{(0)} + p_i^{(0)} = \ell(l)\}$.
6: if $A_i = A_{i-1}$ and $\text{ref\_flag} = \text{true}$ then
7:     Set $\text{ref\_flag} \leftarrow \text{false}$.
8:     Fix active set $A \leftarrow A_i$ and $F \leftarrow A^T$.
9: Initialize SQMR with
10:\[ \begin{bmatrix} r_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} - (g + Wd_i + A^T \lambda_i)^T c + Ad_i \\ \end{bmatrix}, t = \begin{bmatrix} p_i^F \\ p_i^\lambda \end{bmatrix}, q_i = t, r_1 = \|t\|, \theta_1 = 0, \rho_1 = \begin{bmatrix} r_1 \\ s_1 \end{bmatrix}^T q_i, \begin{bmatrix} \tilde{r}_1 \\ \tilde{s}_1 \end{bmatrix} = 0, \text{and } \begin{bmatrix} \tilde{r}_1 \\ \tilde{s}_1 \end{bmatrix} = 0.
11: for $j = 1, 2, 3, \ldots$ do
12:     Compute $t = \begin{bmatrix} z_{j+1}^F \\ z_{j+1}^\lambda \end{bmatrix}$, $\gamma_{j+1} = \|t\|_2 / \tau_j$, $\gamma_{j+1} = 1 / \sqrt{1 + \theta_{j+1}^2}$, and $\tau_{j+1} = \tau_j \theta_{j+1} \gamma_{j+1}$.
13: Compute $\begin{bmatrix} \tilde{r}_j \\ \tilde{s}_j \end{bmatrix} = \gamma_{j+1} \theta_j \begin{bmatrix} \tilde{r}_j \\ \tilde{s}_j \end{bmatrix} + \gamma_{j+1}^2 \alpha_j q_i$.
14: Update $d_{j+1}^F = d_j^F - \frac{\tilde{r}_j}{\tilde{s}_j}$ and $d_{j+1}^A = \ell^A$.
15: if $d_{j+1}^F \geq \ell^F$ then
16:     Update refinement iterate $(d_i^F, d_i^A, \lambda_i+1) \leftarrow (d_{j+1}^F, \ell^A, \bar{\lambda}_{j+1})$; break.
17: end if
18: if $(d_{j+1}^F, \bar{\lambda}_{j+1})$ solves (5.1) then
19:     Compute $\text{return}$ solution $(d_i, \lambda_i)$ with $d_i^F = d_{j+1}^F$, $d_i^A = \ell^A$, and $\lambda_i = \bar{\lambda}_{j+1}$.
20: end if
21: else
22:     Compute $\rho_{j+1} = \begin{bmatrix} r_{j+1} \\ s_{j+1} \end{bmatrix}^T t$, $\beta_{j+1} = \rho_{j+1} / \rho_j$, $q_{j+1} = t + \beta_{j+1} q_i$.
23:     Update $d_i+1 = d_i + p_i$ and $\lambda_i+1 = \lambda_i + p_i^\lambda$.
24:     Set $\text{ref\_flag} \leftarrow \text{true}$.
25: end if
26: end for
27: if $(d_i+1, \lambda_i+1)$ solves (1.1) then
28:     Return optimal solution $(d_i, \lambda_i) = (d_i+1, \lambda_i+1)$.
29: end if
30: end if
31: end for
Appendix B. Details of numerical experiment for random QCQPs. For a QCQP of a specified size and perturbation, the following table lists these quantities: average number of SQP iterations; the number of successfully solved instances, average CPU time (in seconds), and qpOASES pivots for the qpOASES experiments; and the number of successfully solved QPs, average CPU time (in seconds), IQP iterations, qpOASES pivots, and matrix-vector products encountered in total during the iQP experiments.

<table>
<thead>
<tr>
<th>Problem</th>
<th>SQP</th>
<th>qpOASES</th>
<th>iQP</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>m</td>
<td>σ</td>
<td>it</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
<td>0.01</td>
<td>8.2</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
<td>0.05</td>
<td>9.5</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
<td>0.1</td>
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</tr>
<tr>
<td>50</td>
<td>10</td>
<td>0.2</td>
<td>10.5</td>
</tr>
</tbody>
</table>

Continued on next page...
Acknowledgments. The authors would like to thank Frank E. Curtis for sharing his MATLAB implementation p-sqp of the algorithm presented in [8]. We are also grateful to Jorge Nocedal and Andreas Potschka, as well as two anonymous referees, whose comments on the manuscript helped to improve the exposition of the material.

REFERENCES


